ALGEBRA

BOOK I

Teachers' Guide for Class V

Revised Version

Prepared by
NCERT Mathematics Study Groups

National Council of Educational Research and Traini

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PREFACE

For the last ten or twelve years a revolution, popularly known as the 'New Mathematics Revolution' has swept the world. It has involved an expenditure of millions of dollars, roubles, pounds, francs and rupees. What is more important, it has involved millions of man-hours of school teachers, college teachers, educational administrators and some of the foremost mathematicians of modern times. It has changed the attitude to mathematics of billions of children, teachers and parents.

About a hundred projects* have been set up all over the world for developing new curricular materials including text books, teachers guides, charts, models, film strips, films, T. V. lessons etc. and for training teachers in new ideas. In India, the efforts for improving school mathematics have included setting up a panel of distinguished mathematicians under the chairmanship of Prof. Ram Behari, an Editorial board under my chief Editorship, a project under the auspices of the Department of Science Education of the N.C.E.R.T., organisation of six Study Groups at Bangalore, Baroda, Delhi, Jaipur, Jadavpur and Kanpur, a large number of summer institutes for training of teachers and a national conference on school mathematics.**

The six study groups for developing new curricular materials in mathematics were set up in 1966 as a result of the decisions of the conference on mathematics and science education held under the chairmanship of Prof. D. S. Kothari. The coordinating committee of the Mathematics Study Groups decided to assign responsibilities as follows; Primary Mathematics (Class I—IV) and Applications of Mathematics to Kanpur Study Group, Algebra (including Arithmetic) for classes V to VII and VIII to X to Baroda, Delhi and Jaipur groups and Geometry (including coordinate geometry and trigonometry) for classes V to VII and VIII to X to Bangalore and

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*Lockard D Report of the International Clearing House on Curricular Development in Science Education, Maryland University, U.S A.

**Kapur, J N Proceedings of the National Conference on School Mathematics, Mathematical Association of India

Jadavpur groups. The entire work was to be coordinated by the coordinating committee consisting of the six Directors of the Study Groups.

So far the Study Groups have produced the following books :

(i) Handbooks for Teachers of classes I and II
(ii) Algebra text books and their teachers' guides for classes V, VI and VII
(iii) Geonetry tex t books and their teachers' guides for classes V, VI and VIII.

Experimental editions of these books are now being made available for try-outs in some selected schools. The final editions of these books improved in the light of the reactions of students and teachers will be made available for open use by state governments sometime in 1970. The present editions are for restricted use by authorised schools only.

Though Baroda, Delhi and Jaipur Groups worked jointly in writing the earlier versions of algebra books, the final version was prepared by the Jaipur Group. Throughout the preparation of the final version, the Jaipur Group was assisted by Shri P. K. Srinivasan of the Kanpur Study Group

The final version of the geometry text-books has been prepared by the Bangalore Group and that of the geometry teachers' guides has been prepared by the Jadavpur Group.

In algebra, the approach has been through sets, truth sets of open sentences including inequalities and structures of the systems of natural numbers, integers and rational numbers. In geometry, the approach has been through transformations and symmetries. Both the approaches are likely to be new for teachers and therefore all efforts will be made to train the teachers in the new ideas. Even when training programmes are not available, it is hoped that the teachers' handbooks will be sufficiently lucid and clear to provide sufficient help to interested teachers.

Most of the ideas contained in these books have been tried out with many batches of students and the reaction of student and teachers has been of unmitigated enthusiasm. These were also discussed at the National workshop on school mathematics held at I.I.T., Kanpur in December 1968 and attended by representatives of eleven state institutes of education. Their enthusiastic support has been a source of strength to us.

I must take this opportunity of thanking Prof. D. S. Kothari, Chairman of the coordinating committee of all Science and Mathematics Study Groups, Dr. L. S. Chandrakant, former Joint Director NCERT, Dr. S. K. Mitra, former Joint Director NCERT, Dr M. C. Pant, Head of the Department of Science Education NCERT, Shri Rajendra Prasad, Field Officer NCERT, Shri D. Raghavan, Chief, Publication Unit NCERT, Shri N. Chakravartty, Chief Production Officer NCERT, Shri N. A Sriram of Bangalore Government Press for administrative support at all levels without which production and printing of these books would not have been possible.

On the academic side, I would like to thank all the Directors and members of the Study Groups whose names are given elsewhere for their cooperation. I must make special mention of Prof. K. Venkatachalienger whose group prepared the first detailed curriculum of geometry and the first version of the geometry books, of Prof. G. C. Patni for giving valuable leadership to the Jaipur Group during the preparation of the final version of algebra books and of Dr. D. K. Sinha for getting geometry teachers' guides prepared.

I must thank all the managers of the presses for their sincere cooperation.

All the teachers are requested to fill in the proformas given in the teachers' guides and return these to the undersigned.

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SOME REMARKS ON THE TEACHERS' GUIDE

This book is essentially addressed to the teachers and its chief purpose is to help the teachers in picking up the central thread that runs through the topics delineated in the book. The topics in the book, it must have been seen, are far from being a rehash of old ideas which we have hitherto called algebra at the school level. On the contrary, the subject matter dealt with in the book reflects, by and large, the fascinating evolution of mathematics in the last half century. The word 'algebra', it may be mentioned, has been derived from the Arabic word '*al-jebr*' which means the transposition of negative terms in an equation to the either side. The signs of algebra were developed first of all in studying the properties of integers by the ancient Babylonians, Greeks and others. In the early century of the Christian era, the use of operations with symbols arose mainly in India. Later, the development of algebra was carried on by Indian mathematicians such as Aryabhatta, Brahmagupta, Bhaskaracharya and others, who gave the operations with negative integers and zero elements. The book by Diaphantus (250 A.D.?) had a decisive influence on the later development of algebra after the time of European Renaissance (14th Century A.D.) starting from Tartaglia, Fermat, Newton, Euler, Lagrange and others. The next stage in the development of algebra started with Gauss, "the prince of mathematicians" in his great book '*Disquisitiones Arithmeticae*' (a book on Theory of Higher Arithmetic or Theory of Numbers). The celebrated names of fathers of classical algebra are Abel (1802–1829), Galois (1811-1832), Jacobi (1804-1851), Dirichlet (1805-1859), Riemann (1826-1866), Kummer (1810-1893), Kronecker (1823-1897) and Dedekind (1831-1916). The recent abstract and general concepts of present day algebra (called Modern Algebra) starts with pioneers such as Hamilton (1805-1865), Grassman (1809-1977) and Cayley (1821-1895).

Modern Algebra has created algebraic structures of a very general nature such as group, ring, field, vector space, etc. This 'Modern Algebra' has its roots in classical algebra, no doubt, and has as many varied applications to concrete situations as classical algebra has. The introduction of the new relevant concepts in school algebra at present has become very necessary, since we have to make the students understand some unifying concepts such as sets, relations, mappings etc. The present advancement of human knowledge and activity makes it imperative that the students should acquire a proper understanding of these general concepts and the teacher also should have a correct perception of these topics. It is desirable that the teachers should be in a position to interpret the previous knowledge acquired by them in terms of new concepts. The topics in the book have been so framed and emphasised that the ideas of structures recur at a number of appropriate places throughout the book. Further, the modern approach to the study of geometry, as initiated by Felix Klein in his famous 'Erlangen Program', puts a good deal of slant on 'geometric structures' and according to Klein, in the study of geometry, the two structures—algebraic and geometric—are so inextricably interwoven that it is only a fusion of algebraic and geometric methods that can bring to the fore the tenor of thinking in modern mathematics.

In this book, some broad suggestions are offered and they are intended to illuminate difficult parts of the new curriculum and also, to provide the teachers with what is happening beyond the curriculum followed in the text book. This guide has been arranged in chapters corresponding to those in the textbook. In dealing with every chapter, we have talked about

(*i*) the aims and objectives of the chapter,
(*ii*) the method of introducing concepts occurring in different sections of the chapter and touching briefly on their formal aspects in 'general notes' which the teachers may read, if they think that they are necessary,
(*iii*) understanding and skills to be achieved,
(*iv*) class-room and student activities including oral drills, if any,
(*v*) additional and special exercises, if any,
(*vi*) answers to the exercises in the text-book, if necessary,
(*vii*) the whole list of concepts, subconcepts, vocabulary and symbols used in the chapter,

(*viii*) pertinent references at the end of each chapter.

The discussions under the above heads are, inevitably, sometimes more detailed than others. These divisions, even though they exist in their own right, may not always be sharply drawn, and in fact, in many instances they do overlap. It is, therefore, advisable to look upon these separate discussions in a chapter as a single work, for it is then only, we can provide a coherent and developing account of what it is styled as modern mathematics. The same arguments apply to each of the three parts of the books. The three books, though planned for three different classes, are closely related to each other and the topics have immediate relations to each other. It is, therefore, an advantage and, as a matter of fact, the teacher can get the most out of it, if the teacher goes through all the books together with guides before embarking on a course with the Vth class. Besides, what have been stated above, the guide contains one article on general remarks on the teaching of new mathematics together with the teaching of algebra, in particular. A general list of books, journals, articles is given in Appendix II, while Appendix I gives, in a nutshell, contents of some of these books for study and for reference purposes, too.

SOME GENERAL REMARKS ON THE TEACHING OF NEW MATHEMATICS

The only way, it seems, to make the teaching of materials cast in the mould of new mathematics, interesting is that a teacher of mathematics should participate actively in the proceedings of his (or her) class so that the students can discover the mathematical truth by themselves. In other words, the environment of the class should be so conditioned that the students can draw upon the surrounding familiar objects or things so as to get at the truth intuitively and informally. This concept of 'active learning', it may be mentioned, is not a recent coinage of the theory of learning but it dates back to the time of Socrates, who said: 'The ideas should be born in the students' mind and the teacher should act only as midwife'. Subsequently, a number of philosophers such as Kant and Whitehead, among others, also ranged themselves in favour of 'learning that begins with action and perception' and of activities and experiences out of which the right basic concepts can emerge. The classroom situations, local happenings, the experiences acquired in other science classes or elsewhere all together go to make up the totality of what is broadly called the experience of the students as referred to above and if a teacher is to present the new materials as something challenging and exciting, too, he (or she) has to make use of this experience. It is undoubtedly an onerous task, but once a teacher puts a slant to his (or her) teaching in terms of 'experience' 'activity' and 'discovery', he (or she) can not merely stimulate the curiosity of young students but also can have the opportunity for tasting, in part, the joy of discovery and pleasure of excitement of his (or her) students.

It is obvious that no two teachers will explain and illustrate any new topic exactly in the same way, for the experience of one teachers' set of students is very likely to differ from that of the other teachers' set of students. It, therefore, follows that a topic need not be explained in toto as done in this guide, but certainly in keeping with the situation of students and that may even vary from school to school. Thus, the suggestions, contained in this book, serve as illustrations that the teacher may adapt to suit the needs of his (or her) particular class. It is only the teacher of the class who is in a position to know of the particular situation of his (or her) own class and perhaps, none else, not even the framer of the syllabus, the author of the guide books or the text book.

The teachers may profit a good deal from the researches undertaken by eminent psychologists such as Prof. Jean Piaget of Geneva, Prof. Z. P. Dienes of U. K. regarding the formation of concepts such as numbers, point, line, plane etc. and some of these works are referred to the general list (Appendix II) given at the end of the book.

It has already been said in the preceding lines that the aim of a teacher will be to form associations, to link the topics with the things around the students until an use of them becomes effortless. This should form the perspective of the lessons to be framed by the teacher. The teacher may find from time to time that some of the previous topics have not been fully understood and so, he (or she) should immediately revise and reteach them, for unless this reteaching is done now, the students will continue to harbour a wrong notion which may stand in the way of the introduction of a new topic at a later stage. It is to be emphasised that the teacher should never miss such an opportunity. For, otherwise, it is then he (or she) can afford to recapitulate old things before he (or she) launches out to bring in new ideas. It may be suggested that a teacher while teaching a lesson, he (or she) may, whenever possible, introduce some of the concepts of the next lession, so that the teaching of the next lesson becomes more effective.

It is a fundamental principle of learning that the repetition of a topic with a new vocabulary in different situations and contexts is the key to its acquisition by the students and, therefore, the teacher should make it a point to reteach topic as many times as is feasible. Questions and the drills should be intimately associated with the experiences of the students and should be illustrated in situations that have meaning for the students. Unless the teacher builds around the experiences of the pupils, the drills that are not divorced from the situations and activities of life, he (or she) cannot expect his (or her) students to comprehend fully the materials taught.

One can think of several ways of learning mathematics but solving problems is one of the most cardinal tenets of mathematical activity and an essential part of mathematical instruction. In the choice of problems, the teacher will have to exercise his (or her) own discretion and certainly the teacher will choose those which the teacher feels to be of greatest use to the students and of relevance to the topics concerned. Sometimes, the teacher may have to provide simpler examples of some topics and more, for further practice. It has to be remembered that although the textbook contains a number of exercises, the drilling may be oriented by the teacher in such a way as to elicit spontaneously some questions from the students themselves. In fact, the exercises that occur at the end of each chapter should be thoroughly practised or otherwise, until the pupils can solve them without mistakes. It may be pointed out that exercises should not be looked upon as tests. It is believed that an intensive oral drill for sometime followed by an oral practice for another length of time every day will be more effective than that using the whole of each period for oral drill and setting apart one whole period for writing. For a large class, when a teacher finds that most of the students have understood a point, the teacher may divide the students into groups and may assign to each group separate, oral or written exercises. Such grouping may give the teacher an opportunity to work exclusively with slow learners among the students. Some of the unfinished exercises may be assigned for homework. Here, the teacher must bear in mind that the students should never be overloaded with home work of the drilling type, particularly at this elementary stage. It must be borne in our mind that homework, more or less, should be revisionary in nature, so that the students can find time to master what they have learnt in the class. The students should have sufficient practice at home for the lessons done in the class.

In the course of teaching and writing in mathematics, a teacher can hardly afford to do without the use of blackboards. The equipments that are essential for intensive classroom and student activities are plain papers, graph papers, scissors, rulers, blackboards ruled into squares, models made up in cards and other materials, peg-boards, desk calculators, geoboards. It would be excellent if the aids to instruction include some programmed materials (besides textbooks), overhead projectors, transparencies, films and film-strips, structured aids, analogy devices etc.

Finally, it may be summed up that the path to success by teachers lies in an increasing use of the environment, and of teaching aids, as well. But, in the ultimate analysis, a good teacher can always evolve an approach, even with limited resources and media, if the teacher is keen on seeing that the students do learn the notions properly and correctly.

CHAPTER I

SETS—AN INTRODUCTION

I. Aims and objectives of the Chapter :

The main aim of this chapter is (i) to introduce the idea of a set and associated operations, (ii) to enable the students to use the set language. These are necessary for a modern mathematical education.

II. Method of introducing concepts :

General notes: A glance through any curriculum which aims at modernising school mathematics would reveal that the concepts of sets have to be the starting points in any curriculum. If the aim of teaching modern mathematics is to emphasise the structural aspects, then one has to cast out the useless parts of the syllabus and bring in these that not only have an unifying role but can bring us to the frontiers of modern ways of thinking in mathematics. 'A number of experiments have shown that it is possible to use the language, elementary concepts and operations of sets and the concepts of relations and functions at the age of twelve (or even earlier)', so goes one of the recommendations of UNESCO symposium held at Budapest in 1962 on the teaching of mathematics and the last decade has witnessed a number of projects in developed countries of the world trying out with the idea of sets in lower forms of the school, in German Gymnasium and in French Lycee. The reasons for including such a course on sets may be summed up as follows: (i) such ideas are basic to the whole of mathematics, (ii) ideas lead to the sharpening of the understanding of other branches of mathematics, (iii) the symbols are useful for explaining mathematical arguments clearly and concisely, (iv) 'set language' is convenient for and relevant to other branches of mathematics, say, Boolean algebra, data processing, theory of probability and so on.

The theory of sets, as we know it to day, is due to the German mathematician George Cantor (1845-1918). It was in connection with a problem of series that Cantor threw himself wholly to develop a rigorous theory of sets. An interested teacher might go through the historical resume′ given in the treatise of Bourbaki vide, Appendix II. According to Cantor, we can look upon a set as a multitude conceived of by us, as 'one', vide, Vilenkin. Loosely speaking, we can say that a set is a collection, a family, a system, a class, an ensemble etc., and we can use as many as synonyms as possible for the word 'set'. But the definition that would be mathematically tenable should be one in which the elements in a set are all well-defined and the set is not defined by its elements such as a set of all abstract sets, etc. A book dealing with abstract ideas of sets certainly points out the paradoxes that arise from the concepts of such sets. There are many examples that are intimately linked up with some deep questions of logic. It may be a piece of information that the answers to such questions have rendered, as it were, a face lift to the subject and are being abundantly used in subjects such as mathematical logic, electronic computers, automatic control etc. But the understanding of the concept of a set has to be founded on the ample and concrete examples drawn from the surroundings. The teacher has to make use of the experiences of students, be it in school or in any other science class, in play ground or in homes or in the clubs etc. What needs to be emphasised is that we never study a single object as totally unrelated to other objects. If we have a cube, we may count its faces or compare its weight with another. A teacher should also use the language of sets in diverse situations and also as an effective instrument in the teaching of geometry as well as of algebra, for example, the set of equilateral traingles, the set of obtuse angled triangles, the set of trapezia, the set of integral multiples of 6, the solution set of an equation etc. As already mentioned, the study of sets has to be viewed in the perspective of teaching of modern mathematics. An important aspect of this teaching is to mathematize a situation. We can formulate it mathematically provided we can recognise what the *elements* are and the *relations* that exist between them. This requires that we must be able to talk about them which, in turn, necessitates the use of symbols or signs. So the elements and the relations constitute the essentials of mathematics. When we talk about the elements, we really work within the set. But if we want to talk about the same for others, we have to think of something of an association between the elements of this set and the elements of the other set. This leads to what we call the *operations* or strictly speaking, the *operation of mapping of one set* into another and finally, we are led to the algebra of sets. A discussion on subsets and reference set or universal set is, in essence, concerned with the hierarchies of sets. We denote this hierarchy by the inclusion symbol $\subset$ or even by $\subseteq$ which are analogues of the inequality signs $<$ and $\leqslant$.

In any discussion of a problem where certain statements are made about certain objects or a set of objects, these are all in the context of a set which contains all the objects under discussion and this may or may not be mentioned, e.g., when we say amongst our friends that the Hindustan Times, the Hindu, the Statesman are good newspapers, it is clear, without being mentioned, that we are talking about papers published in India. Such a set can be called a *universal set* or a *universe of discourse* for that problem or discussion. Thus under different contexts, there may be different universal sets. It should be clear that the universal set does not mean that it is the set containing every object in the universe, as it is impossible to have something which contains everything.

The properties of sets are studied in the context of universal sets. As mentioned above, that every object of a set under discussion is an element of universal set, the concept of such sets is being introduced here. We have seen that the elements of a set can be of the most varied nature. This means the concepts and theorems of the theory of sets have wide generality and this necessitates the notion of a subset, which appears every time when we have to consider a set, not only in itself but as a part of another larger set. If two sets A and B are such that every element of A is an element of B, then A is defined to be the subset of B and this is represented symbolically as $A \subset B$.

Consider the two sets $C = \{a, b, p, q\}$ and $D = \{p, b, q, a\}$.

Every element of C is an element of D. So $C \subset D$. Also every element of D is an element of C. That is, $D \subset C$. We see that if $C \subset D$ and $D \subset C$, then $C = D$. This idea of equality holds for finite sets. The idea of equality of infinite sets is very complicated and will be taken up subsequently. It is easily seen that a set is a subset of itself and it is called the improper subset. If P is a proper subset of Q, then Q has atleast one element which is not in P.

To distinguish improper subset from proper subset, the symbol $\subseteq$ is sometimes used ; we, however, use $\subset$ for both. When a set has no element, it is called the *null set, empty set* or *void set.* The pupils like the phrase 'empty set' and the suggestive symbol { }. But the more prevalent symbol is ϕ (pronounced 'phi'). To start with, we may introduce the symbol { } for empty set.

If the definition of a subset reads, 'If two sets A and B are such that there is no element of A which is not in B', then a null set can be seen as a subset of all sets.

There is some difficulty in accepting empty set and one-element or unit set as a set, as it affronts the common sense notion of collection. It is a logical necessity that has led to their consideration as sets. We can think of many such sets. We learn to think of sets even of one element.

The symbol for the universal set is U. This should not be confused with the symbol $\cup$ for union.

In applications of mathematics, we often come across elements of a collection of sets that occur in each set or collection. This is how we are led to the idea of forming a new set, called intersection of the given sets or their set *product.* The operation underlying the formation of a new set is often called the intersection which, incidentally, is derived from the fact that when we take the intersection of the sets of points in two geometric figures, we get the intersection of two figures in the ordinary sense of the word. The properties possessed by the operation 'intersection' is similar to those for the operation 'multiplication'. For example, the commutative laws and the associative laws, namely,

$$A \cap B = B \cap A$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

are satisfied. For an empty set ϕ

$$A \cap \phi = \phi$$

which is analogous to the equality $a \,.\, 0=0$. Also

$$A \cap U = A$$

which is analogous to the equality $a \,.\, 1=a$. It may be noted that there are properties of set multiplication that are not analogous to properties of arithmetic multiplication. For example, it B is a subset of A, $B \subset A$, and we have the equality $B \cap A = B$. In particular, $A \cap A = A$

The next operation to be studied is the union of two sets which means forming a new whole out of several sets. Here also, the operation of union of sets has many properties analogous to those of addition of numbers. Thus the commutative and associative laws, namely

$$A \cup B = B \cup A$$

$$A \cup (B \cup C) = (A \cup B) \cup C$$

are satisfied. Further,

$$A \cup \phi = A$$

and $$A \cup U = U,$$

so that we can say if even though the empty set plays the role of zero in the union of sets, the universal set U does not play the role of unity in the addition of numbers. If B is a subset of A, then $B \cup A = A$; in particular $A \cup A = A$.

Given a universal set and a subset, a complementary set can always be formed. The process is called *complementation* and it is a unitary operation like taking square root of numbers. If $U = \{a, b, d, e, p, q, h\}$ and $A = \{d, q, h\}$, then the complement of A with respect to U, is $\{a, b, c, p\}$. It is usual to denote complement of a set A as C (A) or A′. We would prefer A′ to denote the complement of a set.

Now $A \cup A' = U$ and $A \cap A' = \phi$

If in a class, pupils wearing spectacles form a set, pupils not wearing spectacles form the complementary set.

There are some interesting relations where two sets A and B and their complements A′ and B′ with respect to a universal set are considered. Let us make use of some diagrams.

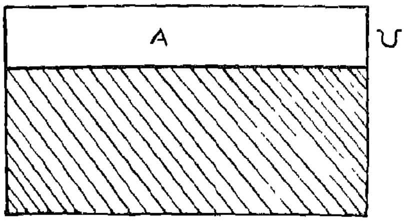

Fig 1.

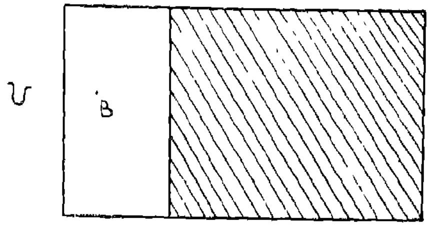

Fig 2

The shaded region in Fig. 1 is A' and that in Fig. 2 is B'.

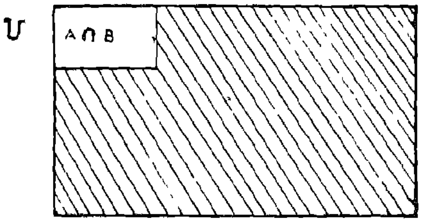

Fig 3

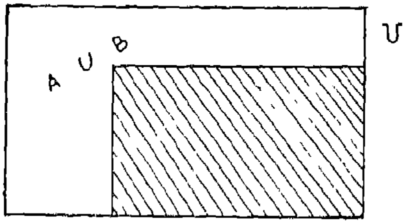

Fig 4.

The shaded region is Fig. 3 is $A' \cup B'$ or $(A \cap B)'$, while that in Fig. 4 is $A' \cap B'$ or $(A \cup B)'$.

So we have $(A \cap B)' = A' \cup B'$

and $(A \cup B)' = A' \cap B'$.

These are called De Morgans' laws.

Finally, a few words about the finite and infinite sets. The idea of finite sets has been introduced first as a collection of objects and then, it has been explained that there are two types of sets, finite sets and infinite sets. The fundamental difference between the two types of sets needs to be explained from the outset. There would be no difficulty in teaching finite sets considered as natural objects. The results that hold for finite sets are extended to those for infinite sets and are used to build up a system of infinite sets. Of course, there is an alternative to this intuitive approach, namely, the formal axiomatic approach to the theory of sets, but it may seem to be difficult for practical use. In this context, perhaps, a skilful teacher may be one who may steer a middle course and who may intermingle the intuitive approach with the formal approach without undermining the correct aspect of mathematics.

SECTION 1-1: *Sets and elements*:

We do not define a set but we clarify our idea of a set by taking examples from the surroundings of the class room. For example, let the students who have odd roll numbers be asked to sit together. Then we get the set of students who have odd roll numbers. We, therefore, see that to get the elements of a set is problem of recognition and decision. In other words, if we describe a set by means of a rule, it may be seen that the rule gives only the numbers of the elements required—no more or no less. It is difficult to decide whether an element belongs to a set if the property by which the element is recognised is not sufficiently definite. For example, if two students of the class are asked to find out the fair-complexioned students of the class, their findings will generally vary. Thus, the set of fair-complexioned students is not well defined as the chosen property is of a subjective nature.

Students may often confuse the part of an element as a member of the set. For example, if T is a set of tulips in a particular vase, then tulips are the members of the set and not the petals, stems etc.

As mentioned in the general notes, the idea of a finite set may be introduced to the students intuitively at this stage. Infiniteness may be confused with largeness. For example, the set of grains of sand on a sea shore is a large set but not an infinite set. This should be clarified to the students.

Concept: Set.

Vocabulary: Set, member of the set, elements of the set.

SECTION 1-2: *Sets and Symbols*:

In the textbook, the set has been represented by writing the elements of the set within braces or by writing the property which characterise the set within braces. For example, if A is set of two digit numbers, then $A=\{10, 11, 12 \ldots\ldots\ldots 99\}$ or A={two digit numbers}

Another way of writing is $A=\{x/x$ is a two digit number$\}$

It is to be noted that a particular element may not appear twice within the braces. If C={consonants in the word Calcutta}, then $C=\{c, l, t\}$

Vocabulary: Listing the elements, curly brackets, braces, infinite sets, belongs to.

Symbols: A, B, C etc. for sets.
a, b, c etc. for elements.
$A=\{b, i, t\} \Longrightarrow$ A is a set consisting of elements b, i, t.
$a \varepsilon A \Longrightarrow$ a is an element of A, a belongs to A
$A \ni a \Longrightarrow$ A contains a

SECTION 1-3: *Venn Diagram*:

After all the events of the sports in a school is finished, the prize-winners are generally encircled by a rope. This is done to separate the members of set of prize-winners from other competitors and visitors. Let us consider the set C={bat, ball, wicket}. How to remember that the set C contains only these objects? We can remember them by trying a rope round them. A representative of this is often used to show sets graphically. This is known a Venn diagram or, strictly speaking, Euler-Venn diagram. At an advanced level, a diagram forms no part of rigorous proof and such proofs in set algebra are conducted in terms of elements and the operations of inclusion, complementation, union and intersection. At an introductory stage, however, Euler-Venn diagrams are very useful to clarify the whole subject. It should be kept in mind that Venn diagrams are (topological) representations and the relative sizes and shapes of the regions drawn are not the important factors. When we represent a set diagramatically, we do not attempt to draw the object or show where they are. We limit ourselves to a drawing of the rope. Thus the set will simply be represented by drawing the rope (closed curve). We do not attempt to draw the objects or show where they are. We agree to represent every object by a point inside or outside the closed curve. No object is represented by a point on the curve. The set of the elements will be represented by the points inside the closed curve, while the elements which do not belong to the set will be represented by points outside the closed curve. We will only represent those points to which we want to draw attention.

Vocabulary: Euler-Venn diagram.

SECTION 1-4: *Equal sets*:

It is to be emphasised that the order in which the elements are listed within the braces does not matter.

Concept: Equality of sets:

Vocabulary: Equal sets, identical sets

Symbol : $A=B \Rightarrow$ the set A is equal to the set B. i.e.
$x \varepsilon A \Leftrightarrow x \varepsilon B$

SECTION 1-5: *Set and their subsets*

The idea of subsets can be made clear to the students by examples taken from the classroom. For example, the set of students of class V is a subset of all students of the school; the set of subsets of class V who can play football is a subset of the set of students of class V. However, it may be possible that all the students can play football form a subset of the set of all the students of class V. Thus we find that 'sub' does not mean smaller. A subset may or may not be equal to the set. In the latter case, we call the subset a proper subset. We can also may that every set is a subset of itself.

When we form the set of students of class, V we choose the elements of the set from the set of all students of the school. The latter set may be said to be the universal set. For example, in algebra, the universal set may be nonnegative rational numbers; in geometry, it is the set of all points in Euclidean space. Any geometric figure is, therefore, a subset of the set of all points in Euclidean space. Again, it may happen that no student of class V can play football. In this case, the set of students of class V who can play football will have no member. This set is called empty set. Thus empty set is also a subset of the set of students of class V. The set of parallelograns is a subset of trapezia; the set of negative integers is a subset of integers. The set of triangles in which the sum of whose angles is different from 180°, the set of quadratic equations having more than two roots etc. are examples of empty sets,

Let us now consider two sets A and B. Let us suppose both of them to be empty. If we are asked the question: "Is ε an element of A?" "Is it an element of B?" Clearly our reply will 'no' to both questions, whatever it may be. So, $A=B$.

Hence, we see that there is one and only one empty set. Sometimes, the confusion may arise in the notation of an empty set, such as {o}, {ϕ}. The correct one, as we know, is just ϕ or { } {o}, {ϕ}. Incidentally, it may be mentioned that the symbol ϕ for the empty set has been derived from the Danish letter ϕ pronouced for 'a'.

If U={ a, b, c } be a universal set, then U has the following subsets:

{ }, {a}, {b}, {c}, {a, b}, {b, c}, {a, b, c}

If U has n elements, then U has 2^n subsets, for each of elements may be selected or discarded,

There is a possibility to confuse the symbols ε and ⊂. It is to be made clear that the symbol preceding ε must represent an element, that following it must be a set; whereas for ⊂ both must be sets.

Concepts:	Subset, universal set, empty set.
Subconcepts:	Proper and improper subsets.
Vocabulary:	Subset, proper and improper subsets, included, not included, empty, void, null, non-empty, nonvoid.
Symbols:	$A \subset B \Rightarrow$ the set A is a subset of B { } or $\phi \Rightarrow$ empty set.

SECTIONS 1-6 and 1-7: *Intersection & Union*:

We shall find that the idea of intersection is needed more often, particularly in geometry where, for example, constructions depend on the intersections of point sets or line sets.

When forming unions and intersections, it is to be remembered that an element is not included twice in a set. If A be a set consisting of 15 elements and B be a set consisting of 10 elements, then their number of elements in $A \cap B$ may range from O to 10 and that of $A \cup B$ from 15 to 25.

Questions that may be discussed in connection with union and intersection are:

Can $A \cup B$ be equal to $A \cap B$?
Can $A \cup B$ ever be a subset of $A \cap B$?
Can $A \cap B$ always be a subset of $A \cup B$?

The various cases may be illustrated by Venn diagrams.

Though Venn diagram is the only useful tool for the beginners to illustrate these operations, they may be told how the intersection and union are represented in the set language:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$
$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

The students may be asked to establish the following results with the help of Venn diagrams:

$$A \cup A = A, \quad A \cap A = A.$$

When B is a proper subset of A, then

$$A \cup B = A, \quad A \cap B = B.$$

Concepts: Intersection, Union.

Subconcepts: Disjoint sets.

Vocabulary: Intersection, union, disjoint sets.

Symbols: $A \cap B \Rightarrow$ A intersection B, i.e., a set of elements common to both A and B.

$A \cup B \Rightarrow$ A union B, i.e., a set of elements either in A or in B.

III. Understanding and skills to be achieved :

The importance of the teaching of sets and its algebra at an early stage hardly needs any reiteration. Nevertheless, the point that requires emphasis is that the idea is to be introduced to the students not as an end in itself but as something that has to be interwoven with the realities of the situations surrounding the students. In other words, one has to abstract from the realities of mathematical situations to which the students should be exposed in order that they may develop some abilities that will sustain their thinking at higher levels.

Although the mathematics at this level has to start from the experiences, the teacher has to capitalise these experiences, to his advantage, to the building up of a mathematical language in written and verbal form, that will make the experiences sufficiently clear and precise.

The students must be in a position to distinguish between the elements of a set i.e., to distinguish green pencils from the blue ones and hence the teacher should make it a point to see if the vocabulary of the set theory and the symbol used are properly understood by the students. At this stage, the students should be able to distinguish parts within whole and if they can do it meaningfully, then one can be sure that they have understood subsets and complementation. If a student can pick out the pencils which are both blue and green, it may be taken that he has understood 'intersection' and if he recognises, that which is blue or green, he has understood 'union'. It may be mentioned that this is how the students may be stimulated to the exercise of his hands and senses in performing these operations with set theory so that algebraic structures, in turn, will unconsciously produce in his mind the 'logical structures,' one of the finest form of which is the Boolean algebra. To summarise the understandings and skills, that students will achieve from this chapter are (i) the students will be able to construct a set of objects with some common characteristics, (ii) the students will have to recognise the elements of a set (iii) the students discover that set membership is independent of special arrangement and a set can be determined by different descriptions (iv) the students intuitively see the distinction between finite and infinite sets and construct examples of them (v) the students, specify a set by listing or by description (vi) the students, recognise the universal set implied in a context and realise that it varies from context to context (vii) the students recognise and construct subsets of a set (viii) the students discover that a set is a subset of itself and that the empty set is the only subset of all sets (ix) the students understand the equality of sets (x) the students will be able to form new sets by set operations 'intersection' and 'union' (xi) the students will have the skill of representing sets and illustrating set operations by Venn diagrams (xiii) the students will understand and be able to make use of the set vocabulary (xiii) the students will be able to use symbols and translate sentences involving them. The exercises given at the end of each section are intended to familiarise the students with the idea of a set, the notation for a set, the different ways of describing a set, to develop the use of Euler-Venn diagrams with the emphasis on two operations of sets, intersection and union.

IV. Classroom and student activities :

The students should be given opportunities to suggest characteristics by which a set is to be defined and to form sets of objects having those characteristics. The students should be led to such situations that they can discover for themselves that the elements of a set must be welldefined and distinct. Universal set, subset, empty set, disjoint set etc. and the set operations 'intersection' and 'union' can be illustrated by taking examples from class room situations. It is for the teacher to find out appropriate characteristic or characteristics to illustrate each of these things; group photos, geometrical models, coloured cards, coloured pencils may also be utilised. A pack of punched cards (which may be obtained from the computer department or suppliers of office equipment) may be used to explain the set operations of intersection and union. Each card has holes punched round the edge and we can cast away some of the holes, replacing them by slots. Let

$X = \{$cards with a hole in the first position$\}$
$Y = \{$cards with a hole in the second position$\}$

We can demonstrate $X \cup Y$ or $X \cap Y$. If, again, we remove the set X from the pack by inserting a needle through holes in the first position and simply lift them out, we are left with cards with a slot in the first position, i.e. a complementary set X'. The following method will also provide a stimulating way of investigating operations on sets. Three sets of cards labelled A, B, C are made and then handed over to the students such that some students got two cards A,B, or B,C but not A and C. The students who have cards labelled A or B or both in their possession are asked to stand. Thus the operation $A \cup B$ is understood. If the students who are holding cards labelled A and B both are asked to stand, then the operation $A \cap B$ is explained.

If the students having in possession two cards labelled A and C are asked to stand, then no student will stand. Thus, we get a disjoint set. Students should not be given formal definitions and be asked to memorise them. Definitions may be arrived at after class discussions with a host of examples in the class and gradual refinement. The students' satisfaction in expressing clearly what they have realised through meaningful experiences in relevant situations should alone determine the time and the mode for fixing up definitions. The students find a lot of enjoyment in representing sets and their relations and operations by diagrams. They should be encouraged to make their own charts and to put them up in the class. This would create a climate for acceptance of ideas, for exercise of originality and development of initiative. Drill should not be of a mechanical character but be motivated by improvement of awareness and need for skill to handle the mathematical situations. Another interesting part of the classroom activities may be the preparation of small workbooks on sets, Venn diagrams and the operations of sets. The teacher may prepare such workbooks in the following pattern:

This is how we list the members of a set

{Pen, pencil, ink}

We enclose them in *curly brackets* and put a *comma* between each of the members.
The........tell us where the set begins and ends.
The........separate the members.

The symbol......stands for ' is a member of ' or 'belongs to'
The symbol...... stands for 'is not a member of' or 'does not belong to'.

The......of two sets A and B is the set of things that are in both set A *and* set B.
The symbol......is for '*intersection*'.
If two sets have no members in common we say that they are....

The teachers may use these workbooks for the students who might have joined late and also for slow learners.

Teaching aids:

(i) Class room situations, (ii) collection of objects by children, (iii) Group photos (iv) models of geometrical figures, (v) coloured cards, (vi) Coloured pencils, (vii) Punched cards, (viii) Black boards (ix) Coloured chalks, (x) Charts.

V. Additional Exercises:

1. In each of the following, list the elements of the set where possible; if you cannot list all the elements, list five elements and then give dots; write φ when there are no elements.

(a) A={The prime numbers between 24 and 33}
(b) B={The odd numbers between 20 and 40}
(c) C={The numbers between 20 and 27 divisible by 7}
(d) D={The odd numbers}
(*e*) E={The divisors of 40}
(f) F={The multiples of 40}
(g) G={The numbers divisible by 6, but not divisible by 3}
(h) H={The square numbers}
(i) I={Odd numbers divisible by 2}

2. For each of the following sets (left-hand column) write down the letter of the set in the right-hand column which is equal to it.

(a) {1, 2, 3, 4}
(b) {2, 4, 6, 8}
(c) {0, 1, 2, 3}
(d) {All the odd numbers greater than 7 but less than 11}
(e) {16, 9, 4, 1}
(f) {1, x, x}
(g) {numbers obtained by adding any pair of numbers in the set 2, 3, 4}
(h) {All the factors of 10 which are larger than 10}

A={1, 2, 3}
B={9}
C={1, 2, 3, 4, 5, 6, 7}
D={5, 6, 7}
E={The even numbers greater than 1 but less than 10}
F={φ}
G={the first four natural numbers}
H={1, 2, 3, 0}
I={All the positive integers less than 20 which are perfect squares}
J={x^2, x, 1}

3. The following diagrams represent two sets A and B described as follows. Write the number denoting the sets and the corresponding letter denoting the figure which correctly represents the sets in each case.

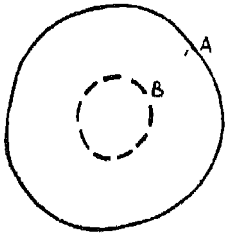

Fig. 5(a)

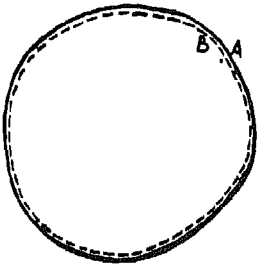

Fig. 5(b)

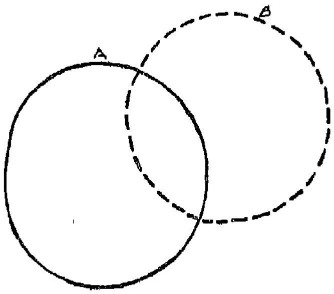

Fig. 5(c)

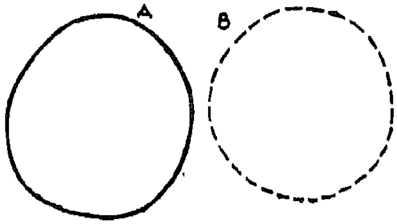

Fig. 5(d)

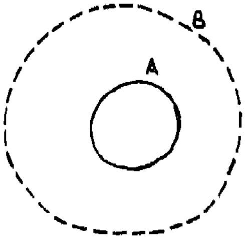

Fig 5(e)

(1) A = {even numbers} B = {odd numbers}
(2) A = {living things} B = {plants}
(3) A = {Divisors of 24} B = {Divisors of 36}
(4) A = {The set of letters of the word 'pot'}
B = {The set of letters of the word 'top'}
(5) A = {rectangles} B = {quadrilaterals}
(6) A = {students of class V} B = {students of class VII}

4. Of 20 boys in a class, 10 play football, 14 play criket (10+20) is more than 20, what is the explanation? Represent this in a Venn diagram.

5. Draw a Venn diagram for the U of all quadrilaterals and the subsets.

P = {parallelogram}
T = {Trapezium}
R = {rectangle}

6. Simplify:

(a) $A \cup A$, (b) $A \cap A$, (c) $A \cup U$, (d) $A \cap U$, (e) $A \cup \phi$, (f) $A \cap \phi$ (g) $\phi \cup U$, (h) $\phi \cap U$.

7. What can you conclude from each of the diagrams in which A, B, C represent straight lines? Draw A, B, C in the plane of your exercise book.

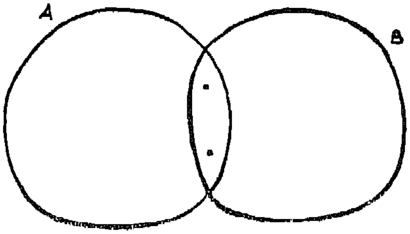

Fig. 6

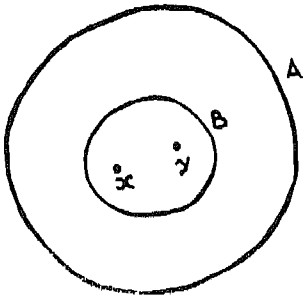

Fig. 7

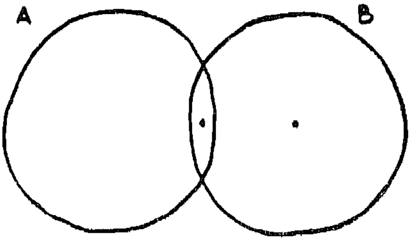

Fig. 8

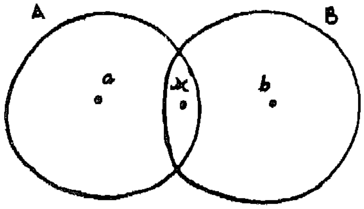

Fig 9

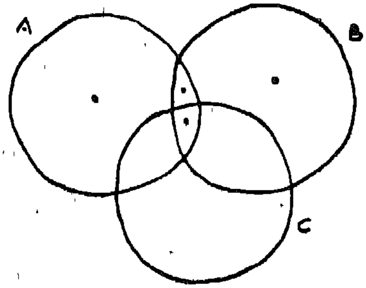

Fig. 10

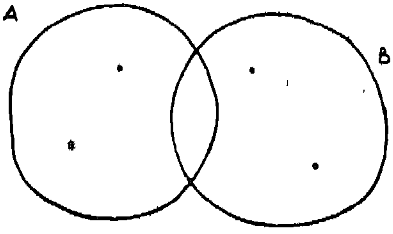

Fig 11

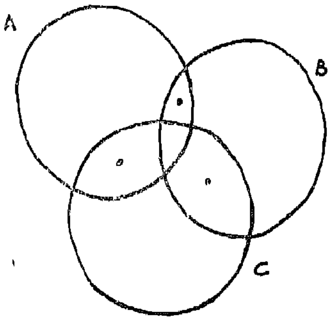

Fig. 12

8. Illustrate the intersection set for the pairs of sets in the following figures.

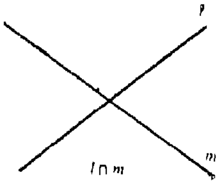

Fig. 13

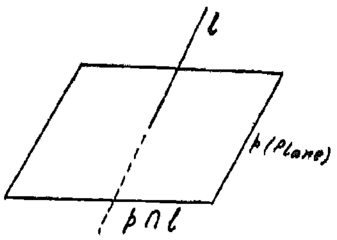

Fig. 14

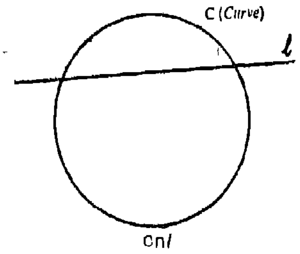

Fig 15

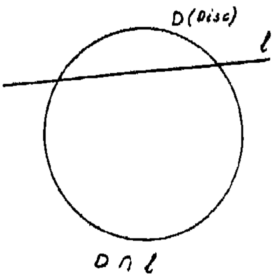

Fig. 16

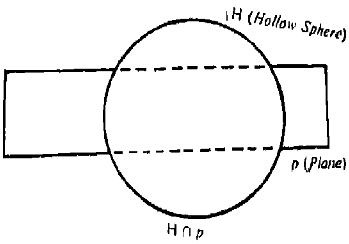

Fig. 17

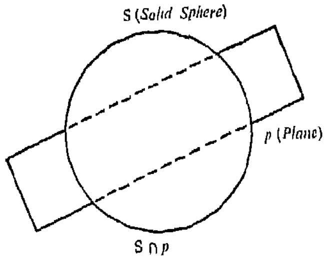

Fig. 18

VII. List of concepts, subconcepts, vocabulary and symbols used in this chapter.

Concepts: Set, equality of sets, subset, universal set, empty set, intersection, union.

Subconcepts: Proper and improper subsets, disjoint sets.

Vocabulary: Set, member of the set, elements of the set, listing the elements, curly brackets, braces, infinite sets, belongs to, Euler-Venn diagram, equal sets, identical sets, subset, proper and improper subsets, included, not included, empty, void, null, non-empty, nonvoid, intersection, union, disjoint sets.

Symbols: A, B, C etc, for sets; a, b, c etc. for elements.
A=> {b, i, t}=> A is a set of consisting of elements b, i, t
a ε A=> a is an element of A, a belongs to A
A ∍ a=> A contains a

A=B=> the set A is equal to the set B.
A⊂B=> the set A is a subset of B.
{ } or ϕ=>empty set.
A∩B=> A intersection B, i.e., a set of elements common to both A and B.
A∪B=> A union B, i.e., a set of elements either in A or in B.

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CHAPTER II

THE SET OF WHOLE NUMBERS : W

1. Aims and objectives of the chapter

One of the aims of this chapter is to build upon the knowledge of sets gained in the previous chapter, the idea of counting numbers or natural numbers, as we often say and the essential pre-requisites for this are the ideas of *equipotent* (or equal) sets, cardinal numbers, ordinal numbers and so this chapter deals with these preliminary notions. This chapter also provides an opportunity to study the basic operations viz., addition, subtraction, multiplication and division on the system of whole (or natural) numbers. Finally, the chapter sets out to acquaint the students with some of the fascinating patterns of natural numbers, the prime numbers, H.C.F. and L.C.M. etc. of numbers, that constitute the rudimentary part of higher arithmetic (otherwise, called the Theory of Numbers).

2. Method of introducting concepts

General notes: In this section, we propose to give in brief the formal aspect of the whole numbers. Even though the invention of number is a fact of history, there is no denying the fact that the numbers where first used for counting sets of objects and the first counting numbers are 1, 2, 3......In fact, this is, by and large, an approach followed in the textbook. The process of counting is nothing but to set up what we call one-to-one correspondence between the set of objects and the set (or strictly, a subset) of the natural numbers particularly, positive integers, to be done later on. For example, we may consider A to be the set of students in a class and a set B as the set of their roll numbers. For every roll number, there is only one student and every student only one roll number. This may be taken as obvious in the sense that there are as many students as the roll numbers. A similar situation may be thought of between the set of students present in a class and the set of occupied desks in a class. It may not be out of place to mention that we often use the word 'mapping' in lieu of correspondence. But, then, what is mapping? Let A and B two sets and let 'f' be a rule that associates a unique element of B with an element of A. Then we say that f is a *mapping of A into B* and write "f: A→B". If an element $a \varepsilon A$, the element in B associated with 'a' is denoted by $f(a)$ and is called the image of a under f. The set A is called the *domain of definition of* 'f' or simply the domain of f and the set of images in B is called the range of f. Let us consider a few illustrations. Let A={1, 2, 3, 4, 5, 6, 7} and B={1, 3, 5, 7, 9}. We can define a mapping f: A→B by specifying the image under f of each element of A as follows:

$f(1)=3$
$f(2)=5$
$f(3)=7$
$f(4)=3$
$f(5)=3$
$f(6)=9$
$f(7)=5$

Clearly, here the domain of f is A while the range of f is the set {3, 5, 7, 9} which is obviously a subset of B but not the entire set B. If, however, we define f as:

$f(1)=3$ $f(2)=5$ $f(3)=7$
$f(4)=3$ $f(5)=3$ $f(6)=9$
$f(7)=1$

then the domain of f is the set A and the range of f is the whole of the set B. This type of mapping is called an "onto" mapping. This, if f: A→B and if the range of f consists of the whole set of B, then we may say that f is a *mapping of A onto B* we write f: A onto B. It may be noted that in an "onto" mapping, every element of B must be the image under f of atleast one element of A. The teacher may find it for himself that every "onto" mapping is an "into" mapping, while the converse is not necessarily true.

Now, several element of A may correspond to one of B, but an element of A corresponds to just one element of B and we may speak of this mapping as *many to one correspondence.* A one to one correspondence between the sets A and B is, therefore, a rule by means of which, we can pair off the elements of A against those of B, and each corresponds to unique element of the other set. Evidently, one to one correspondence is a mapping that is *onto* and *one to one* (or a *bijective mapping* or a *bijection*). On the contrary, a mapping that is *one-one* but *not onto* does not yield a one to one correspondence between A and the whole of B but does so between

A and a subset of B. Like the concept of mapping, the concept of relation plays a very fundamental role in various areas of mathematics and other subjects. This idea is relevant to the notion of equivalence classes which will be necessary in this chapter. In common parlance, the relations are often expressed by sentences such as

Krishna is the 'sister of Tarun',
John is the 'brother of' Margarette,
Bimal is a 'friend of' Madhu, etc.

In mathematics, we say 2 is the factor of 4, line n is parallel to m, etc. Let us now consider the set of children.

S = { Krishna, John, Selim, Tarun, Margarette, Rabeya }
where we find

Krishna is the sister of Tarun
Margarette is the sister of John
Rabeya is the sister of Selim

Symbolically, we can represent this by

T = { (x, y): x is the sister of y, x, y ε S}

Obviously, we cannot interchange (x, y) by (y, x) because it may mean that y is the sister of x which is not true. So, the above relation on S is the ordered pair (x, y). We may, therefore, define relation as a set of ordered pairs. It is clear that when the relation is specified the ordered pairs are also known and conversely.

Let us now consider the set A = { 1, 2, 3 }. Then the cartesian product of the set A × A is given by the ordered pair of elements.

{ (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3) }
If we now take a subset of A × A given by { (1, 1), (2, 2), (3, 3) }, we see that this subset of ordered pairs defines the "is equal to" relation. If, however, we take a second subset of A × A given by
{ (1, 3), (1, 2), (2, 3) }
we find that this subset describes the relation 'is less than'. Similarly, another subset

{ (2, 1), (3, 1), (3, 2) } of A × A

describes the relation 'is greater than'.

We can also represent a relation between the elements x, y of a set S by xRy. A relation R is set to be reflexive on a set S if for each element $x \in S$ it is true that xRx. For example, the relation 'is equal to' when applied to the set of numbers is reflexive since $x=x$ for every number. On the contrary, the relation 'is the father of' applied to people is not reflexive, for no person is the father of himself. A relation R is said to be symmetric on a set S if wherever xRy for every $x, y \in S$, we have yRx. If x and y are two numbers and if $x=y$ then $y=x$ and so 'is equal to' is a symmetric relation on the set of numbers. Its example in common usage is given by 'is married to'. Finally, a relation R is said to be transitive on a set S, if whenever, xRy and yRz for $x, y\ z \in S$, we have xRz. The relation 'is equal to' on the set of numbers is a transitive relation. Another example of transitive relation is given by 'is greater than' on the same set. For, if $x>y$ and $y>z$ then $x>z$.

A relation which is reflexive, symmetric and transitive is called an equivalence relation and generally is denoted by $\simeq$.

Let us consider on the set S of students in a particular school, the relation R defined by 'is the class-mate of'. This relation is reflexive for xRx, $x \in S$. It is symmetric for if xRy then yRx, $x, y \in S$

Also the relation is transitive for if xRy and yRz then xRz, $x, y, z \in S$. So the relation R is an equivalence relation which, divides the set of students in the school into separate (mutually disjoint) classes. Thus an equivalence relation on a set S partitions the elements of S into classes which form mutually disjoint sub-sets whose union is the whole set S. Such classes are called *equivalence classes*.

As explained in the textbook, we say that the two sets are similar (or equivalent or equipotent) if they are in one-to-one correspondence. So in the language of mapping a set A is equivalent to a set B if there exists a one-to-one mapping of A onto B. The relation' A is equipotent to B' which is denoted by E.q (A, B) is an equivalence relation. First, it can be verified that the one-to-one correspondence relation is reflexive since the elements of the set A $\{a_1, a_2, a_3 \ldots\}$ can be put into one-to-one correspondence with itself i.e. $a_1 \leftrightarrow a_1$, $a_2 \leftrightarrow a_2, \ldots$ Secondly, from the very definition it follows that the relation is symmetric. That is to say if $a \in A$ corresponds to $b \in B$ then $b \in B$ corresponds to $a \in A$ as in Fig. 19.

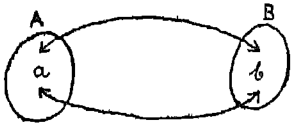


Fig. 19

Finally, let us consider the sets A, B, C such that $a \varepsilon A$, $b \varepsilon B$, $c \varepsilon C$ let $a \leftrightarrow b$, $b \leftrightarrow c$ then from $a \leftrightarrow b \leftrightarrow c$ we have $a \leftrightarrow c$ Hence, one-to-one correspondence of A to B and that of B to C imply the one-to-one correspondence of A to C as in the next figure.

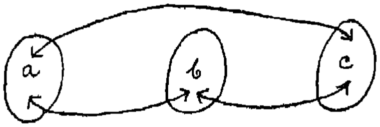


Fig. 20

In otherwords, the relation is transitive. So, Eq (A, B) is an equivalence relation. Accordingly a set consisting of all sets may be partitioned into mutually disjoint classes with respect to this relation. These equivalence classes thus obtained are called cardinal numbers. When two sets can be put into one-to-one correspondence (by some mapping) we say that they have the same number of objects or have the same cardinal number. For example, sets consisting of single objects only have the common property of oneness and we say that the sets are associated with the number we call one. Similarly sets consisting of a pair of elements only are associated with the number we call two, and so on. More clearly, a student uses the set consisting of the little finger along to represent the number one. He may use the set consisting of the little finger and the ring-finger to represent the number two and so on. It can be explained in the following way also. When a student counts "one, two, three, four", he sets up a one-to-one correspondence between the objects he is counting and the sets of number-names one, two, three etc. That is the first object is made to correspond to the set consisting of the single word one. The first two objects are matched with the set consisting of the two words 'one and two' and so on. When a student counts the number four he sets up a one-to-one correspondence between the set of objects and the set of number-names {1, 2, 3, 4}. In other words, the last number-name used in counting the given set is the cardinal number of the set. It was the German mathematician George Cantor who introduced the idea of cardinal numbers to generalise the notion of a number of elements of sets. The cardinal number of a set A is denoted by card (A). It follows from the definition that two sets A and B are similar if and only if card (A) = card (B). A mathematical object 'a' is a cardinal number if there exists a set A such that a = card (A). The operation on cardinals a and b such that a = card (A) and b = card (B) are as follows:

(*i*) The sum of a and b is the cardinal
$$a+b=\text{card }(A\cup B),\quad A\cap B=\phi,$$
(*ii*) The difference of a and b is the cardinal
$$a-b=\text{card }(A-B),\ \text{when }B\subset A$$
(*iii*) The product of a and b is the cardinal.
$$ab=\text{card }(A\times B)$$

A finite cardinal is called a natural number. A non-empty set is finite if its cardinal number is one of the cardinal numbers, 1, 2, 3, and a set which is not empty or finite is said to be infinite. It may not be out of place to mention here that the fundamental system of natural numbers (or counting numbers) forms a vital part of the researches on numbers by the distinguished mathematicians of the nineteenth century, namely, G. Peana (1889—), R. Dedekind (1831—1916) and G. Cantor (1845—1918). Needless to add, we use the sequence of symbols 1, 2, 3, to count the number of object in a set and this is called a cardinal use of the natural number system and this is also used to identify a particular number of the sequence, which is called an *ordinal use* of the number. We can also define a finite set S with 'n' number of elements if its elements can be placed into one-to-one correspondence with the finite set {1, 2, 3,} of the natural number system. On the otherhand, the definition of infinite set is not so obvious. The cardinal number of the set of all natural numbers is denoted by α_0 and the set is called a *denumerable* set. We have seen that the set of natural numbers is not just a random collection of elements. This means, for example, we can always speak of different natural numbers p and q either that $p>q$ or that $q>p$.

The addition in terms of cardinals, means the determination of the cardinal number of a union set. In the language of mapping, addition maps the set of ordered pairs of numbers into the set of whole numbers. For example, this special kind of mapping, maps the ordered pair (2, 3) to the single number 5 and we write this $(2, 3)\xrightarrow{+}5$ or $2+3=5$. We may note, incidentally, that we have chosen ordered pairs from the system of natural numbers and the numbers assigned as the sum to each ordered pair is selected from the same set of natural numbers. We may, therefore, observe that addition is a binary operation defined on the system of natural numbers, for binary operation is a mapping which assigns to each ordered pair of objects in a set, another object selected from the same set. Further, the operation of addition is a many-to-one mapping, for different ordered pairs are all mapped into the same image, namely, the added pairs (1, 4), (2, 3), (3, 2) and (4, 1) are all mapped into the same image 5. Moreover, an ordered pair like (1, 4) and the pair (4, 1) obtained by interchanging 1 and 4, have the same image and we can, therefore, write $4+1=1+4$. This is indeed, true for all natural numbers and it, therefore, follows that for natural numbers a and b, $a+b=b+a$. This is known as the commutative law of addition or as we say, addition of natural numbers is a commutative operation. Let us now extend this operation of addition to these numbers and we are ultimately led to $(a+b)+c=a+(b+c)$ where a, b, c are any natural numbers. This is known as *associative law of addition* or we may say that the addition of natural numbers is an associative operation. We may now go on extending the above two laws to a finite number of natural numbers. The next operation, namely, multiplication of two numbers a and b can be defined as in the textbook in terms of the product of finite cardinals of a and b and is denoted by (a,b). In the language of mapping, as in addition, multiplication maps any ordered pair to a whole number, for example (2, 3) to 6 and we write this as $(2,3)\longrightarrow 6$.

That the commutative and associative laws hold even for multiplication, can be shown easily. There is another important, but common characteristic of addition and multiplication of natural numbers. It is known as the distributive law and it expresses the fact that multiplication is distributive with respect to addition i.e., $a(b+c)=ab+ac$. Summing up, we are now in a position to state that a set of elements, called natural numbers, together with two binary operations $+$ and $.$, satisfy the following properties:

N 1. If a and b are in N, then $a+b=b+a$

N 2. If a and b are in N, then $a \, . \, b=b \, . \, a$

N 3. If a, b, c are in N, then $(a+b)+c=a+(b+c)$

N 4. If a, b, c are in N, then $(a+b). \, c=a.c+b.c)$

N 5. If a, N a, b, c, are in N, then. $a \, . \, (b+c)=a \, . \, b+a \, . \, c$

The different systems of numberings use a group of symbols to express the numbers. For example the decimal system uses the ten Hindu-Arabic symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. But whatever the size of the actual collections used, it is this fundamental system of collection which enables us to deal adequately with the large numbers. Otherwise, a totally different symbol would be required for each different number.

Every number is made up from a group of digits and it is the relative positioning of those digits—each to the other—which gives the identity of that number. For instance, the number 1234, in the decimal system, is made up of one thousand, two hundreds, three tens and four ones, and is really a shorthand form of the expression.

$$(1 \times 1000)+(2 \times 100)+(3 \times 10)+(4 \times 1)$$

Since it is easier and more convenient to use exponential notation in discussing place value systems, we review exponents briefly at this time. The idea of exponent is a convention in notation that we adopted by definition. Just as, for convenience, we agreed to write 5n instead of $n+n+n+n+n$; we agree to write $10 \times 10=10^2$, $10 \times 10 \times 10=10^2 \times 10=10^3$, $10 \times 10 \times 10 \times 10=10^3 \times 10=10^4$. The numerals 2, 3, 4 are called exponents and the numerals '10' is called the base. With this notation we can write

$$1234=(1 \times 1000)+(2 \times 100)+(3 \times 10)+(4 \times 1)$$
$$=(1 \times 10^3)+(2 \times 10^2)+(3 \times 10)+4 \times 1$$

It is worth mentioning that Diophantus, the great Alexandriann Mathematician (3rd Century A.D.) called the third power a 'cube', the fourth power a 'power-power', the fifth power a 'power-cube' and so on. Next the Italian Bombelli, who, when using 1 for the unknown, used 2 for its square, 3 for its cube and so on, but Victa improved on this with his a quadrature or a quad, a cube or a cub. etc. The present method of exponents is used by Rene Descartes to show the power to which a quantity is raised.

We now proceed to examine some properties of whole numbers. We have seen that multiplication maps any ordered pair to a whole number. For example the ordered pair (3, 8)—→24, i.e. $3 \times 8 = 24$. The numbers 3, 8 are called divisors or factors of 24. In turn of 24 is called a multiple of 3, 8. In general we may say if a, b and c are numbers (strictly speaking integers) different from zero such that $c = ab$ then a, b are called divisors of factors of c, and c is called a multiple of a and b. We also see that multiplication maps each of the ordered pairs (1, 24) (2, 12), (3, 8), (4, 6) to 24. That is, { 1, 2, 3, 4, 6, 8, 12, 24 is the set of divisor 24. But the set of divisor of 5 is { 1, 5 } From the examples considered above, we may conclude that if *n* belongs to the set of whole number then 1 and *n* are factors of since $n = n.1$. We, thus get that (1) the number one has only one factor 1, (2) there are numbers which have only two factors, namely, the number itself and 1,
(3) there are numbers which have more than these two factors.

When a number has only two factors, the number itself and one, it is called a *prime number* i.e., an integer $p > 1$ is a prime if its divisors are $+1$, $+p$. An integer different from 1 which is not a prime is called a composite. The number one for its special characteristic belongs neither to the set of primes nor to the set of composite numbers. Thus the system of whole numbers may be classified as shown below:

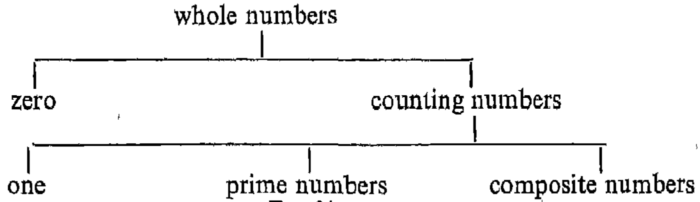


Fig. 21

The whole numbers which are multiples of 2 are called even numbers, the numbers which are not even are called odd numbers. It is worth mentioning here that Pythagoras was one of the first mathematicians to class all numbers as even or odd, but he gave the name 'guomon' to all odd numbers.

It is easy to construct a table of primes upto a moderate limit N, by a procedure known as the 'Sieve of Eratosthemes' devised by the Alexandrian mathematician Eratosthemes (270 B.C.—190 B.C.). All numbers from 2 to N are written down and is then subjected to a process of elimination as follows: all even numbers greater than 2 and all numbers ending in 5 (other than 5) are crossed out. Then, in turn, every third number after 3, every seventh number after 7, every eleventh number after 11 and so on are crossed out. When the process is complete the numbers remaining uncrossed are the primes. We may also find the primes by eliminating the composite numbers. There are a few tests of divisibility which enable one to tell by inspection whether a given number is divisible by few first numbers.

Divisibility by 2. A number is divisible by 2 if and only if the units digit of its numeral is even. The reason for this is that every power of 10 except 10^0 is divisible by 2. Hence the number is divisible by 2 if and only if units digit of its numeral is divisible by 2.

Divisibility by 4, 8. Since every power of 10 except 10 and 10^0 is divisible by 4 and every power of 10 except 10, 10, 10 is divisible by 8, therefore, a number is divisible by 4 if the last two digits when taken together is divisible by 4 and a number is divisible by 8, if the last three digits taken together is a multiple of 8.

Divisibility by 5, 10. A number is divisible by 5 if and only if the units digit of its numeral is 0 or 5 and is divisible of 10 if the units digit is zero.

Divisibility by 3,9. Every power of 10 except 10^0 can be expressed as a sum of a multiple of 9 i.e. of 3 and one. Therefore, the number will be a multiple of 9 i.e. of 3 plus the sum of the digits. Hence the number is divisible by 3 or 9 according as the sum of the digits is divisible by 3 or 9.

Divisibility by 11. Since $10 = 11 - 1$, 10^n can be expressed as a multiple of 11 plus or minus one according as n is even or odd.

Therefore, the number is a multiple of 11 plus the total of the digits in the even places minus the total of the digits in the odd places. So a number is divisible by 11 if the differences of the totals of alternate digits gives a multiple of 11 or 0.

By using the above tests of divisibility, we can obtain the tests for divisibility by the numbers which are products of these numbers (2, 3, 4, 8, 9, 11).

The problem of finding the prime numbers is, in general, tedious. As a matter of fact, the concept of prime and composite numbers is one of the cornerstones in the study of higher arithmetic which often goes by the name 'Theory of Numbers'. Some attempts, though unsuccessful, have been made to find a simple algebraic formula that will yield only prime numbers. For instance, Fermat gave the formula for a prime number as $F_n = 2^{2^n}+1$ for all integral values n. Although we get the primes 5, 17, 257 and 65537 for 1, 2, 3, and 4, the Swiss mathematician L. Euler proved that the result $4294967297 = 641 \times 6700417$ is not prime for $n=5$. There is another formula that yields primes such as $n^2 - n + 41$ upto $n \leqq 40$. An interesting feature of prime numbers is that the number is infinite and the proof of the result that follows is due to Euclid. Let 2, 3, 5,........be the aggregate of primes upto p and let q=2, 3, 5........p+1. Then q is not divisible by any of the numbers 2, 3, 5, p. It is, therefore, either prime, or divisible by a prime between p and q. In either case there is a prime greater than p, which proves that the number of primes is infinite.

We have seen that the set of whole numbers greater than 1 is partitioned into two disjoint sets, the set consisting of the primes and the set consisting of the composites. The primes are, in a sense, the building blocks of the composites. Consider the integer 72. We can think $72 = 9.8 = 3.3.2.4 = 3.3.2.2.2 = 3^2.2^3$. We can also think $72 = 6.12 = 2.3.12 = 2.3.3.4 = 2.3.3.2.2 = 2.2.2.3.3 = 2^3.3^2$. We have looked at the factorisation of 72 into prime factors in two different ways but we have arrived at a unique factorisation except for order of the factors. This result may be stated as follows: A number can be expressed as a product of primes in one way only, apart from rearrangement of factors. This is known as Fundamental theorem of Arithmetic due to great Carl Friendench Gauss. Next we take up the question of 'Highest common factor' and 'Least common multiple'. A set of composite numbers may have factors common. The largest of this set is called the Highest common factor. H.C.F. can be determined by listing the factor of each and by using a complete factorisation of each number. Another method the process of repeated division called 'Euclidean algorithm' is simpler and useful specially for larger numbers. It appeared in Euclid's 'Elements' about 2300 years ago. The interested reader may go through Hardy's Pure Mathematics'. This explains the method of finding L.C.M. by a complete factorisation. Another approach to the operation of finding H.C F. and L.C.M. is to apply the theory of sets. Let us consider the problem of finding the H.C.F. or L.C.M. of 24 and 36. The set of prime factors of 24 is { 2, 2, 2, 3 } and of 36 is { 2, 2, 3, 3 } H.C.F. is the intersection of these two sets and L.C.M. is the union. The intersection of the sets is { 2, 2, 3 }. Hence H.C.F. is 12. The union of the sets is { 2, 2, 2, 3, 3 }. So L.C.M. is 72. Unfortunately in this simple process we encounter a difficulty. In a set, the elements are taken once only. Thus the set { 2, 2, 2, 3 } is actually the set { 2, 3 } and the set { 2, 2, 3, 3 } is also { 2, 3 }. This invalidates the calculation of H.C.F. and L.C.M. However, the difficulty may be overcome if we take set of divisors instead of prime factors. If D (24) and D (36) represent the set of divisors of 24 and 36, respectively, then the set of common divisors is D (24) ∩ D (36)

{ 1, 2, 3, 4, 6, 12 }

H.C.F. is the greatest of these elements which is 12.

Let us now apply Euclid's Algorithm to find the H.C.F.

Let us consider the numbers 18, 60.

D (18)={ 1, 2, 3, 6, 9, 18 }

D (60)={ 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60 }

D (18)∩D (60)={ 1, 2, 3, 6 }

E.C.F. of 18 and 60 is 6.

The difference of the members is 42

D (42)={ 1, 2, 3, 6, 7, 14, 21, 42 }

D (42)∩D (18)={ 1, 2, 3, 6 }

$D(42) \cap D(60) = \{1, 2, 3, 6\}$

$D(18) \cap D(42) \cap D(60) \quad \{1, 2, 3, 6\}$

The difference of 18 and 42 is 24.

$D(24) = \{1, 2, 3, 4, 6, 8, 12, 24\}$

$D(18) \cap D(24) \cap D(42) \cap D(60) = \{1, 2, 3, 6\}$

The difference of 18 and 24 is 6.

$D(6) = \{1, 2, 3, 6\}$

$D(6) \cap D(18) \cap \quad = \quad D(6)$

Thus we find that D (6) is the intersection of the set of divisors of of the original numbers and all the differences. It follows, therefore, that the H.C.F. of all the differences and of the two original numbers is 6. For finding the H.C.F. or L.C.M. of two numbers we can use the property that the product of two numbers is equal to the product of their H.C.F. and L.C.M. This can be esaily shown as follows.

Let h, l be the respective H.C.F. and L.C.M. of the given numbers a, b. Since l is the L.C.M. then l is the least element of the set

$\{ x(ab \div p) : x \in N \}$

$\because ab \div h = b$

or $ab = hl$

2.1 **Comparison of sets:** As new ideas, namely, one-to-one correspondence and equivalent sets, are going to be introduced here to the students assuming that they have elementary ideas of sets, it seems desirable to start with a brief and quick discussion of sets with which the students are already supposed to be acquainted. The teacher should consider some more stories by way of examples, how primitive man developed the idea of counting number by matching objects (or elements) in one set with objects in another set. Sometimes such stories may help in stimulating the curiosity of students. In this connection, the well-known stories about the cavemen with pebbles or the counting of number of ships with stones or how to keep count of the number of wild animals a primitive can used to kill by putting a mark on a stick for each animal. The teacher should make it clear to the students that in each of these cases, there is an one-to-one correspondence between two sets. The teacher should consider some more examples as given inthe text book. After explaining one-to-one correspondence clearly, it is important to mention that when any two sets have their elements in one-to-one correspondence one set has *as many* elements (finite) as in the other. The teacher may then introduce the useful idea of equivalent sets as given in the text book. The following types of examples may be considered.

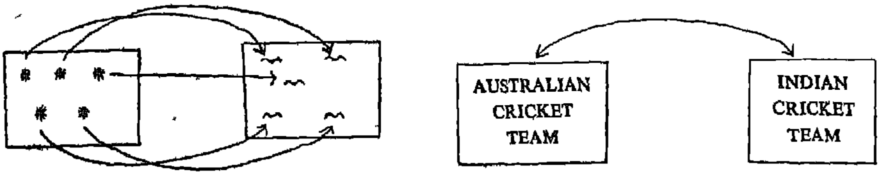


Fig 22

Concepts: The following are the new concepts to be achieved:

(*i*) One-to-one correspondence.

(*ii*) Equivalent set.

2.2. The whole numbers: The teacher should proceed with the counting numbers (or whole numbers) from the knowledge of equivalent sets. The fact, that two sets are equivalent means, there are as many elements in one set as in the other, should be mentioned here again. Then it may be explained to the student that the statement 'two sets are equivalent' can be replaced by the statement. 'Two sets have the same cardinal number'. We have seen that one-to-one correspondence shows that there are as many elements in one set 'as in the other set'. Now let us try to answer the question how many there are in a set? For this purpose we take a standard set of symbols { 1, 2, 3, } and we give the elements the names one, two, three etc.

The following types of examples may be considered:

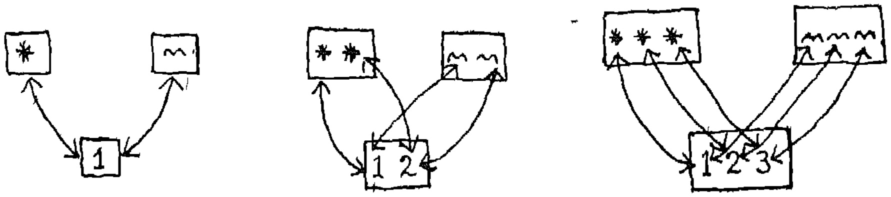


Fig. 23

The point that one, two, three, etc. which we call numbers, are the common properties of equivalent sets, should be carefully explained. The distinction that the number is a concept and numeral is a symbol or representation of the same is to be explained carefully.

The order relation in the set of natural numbers and the idea of the set of whole numbers may be explained as in the text book with some more examples. The teacher may mention that the number one and zero are special numbers in several ways. For example, starting from the natural number, 1, we successively obtain all the natural numbers by adding 1, to the natural number obtained at any stage etc. Zero is not an element of the set of counting numbers, it is a number of the set of whole numbers but most of the time we use it according to the rules of the counting numbers and in a sense it is used to count etc. To explain ordinal number the teacher may consider as an example the students in the class and their roll numbers.

Concepts : The following new concepts have been introduced in this articles:

(*i*) Cardinal number
(*ii*) Counting numbers or natural numbers
(*iii*) Ordinal number
(*iv*) Set of whole numbers

Vocabulary :

(*i*) Cardinal number
(*ii*) Natural number
(*iii*) Ordinal number

2.3. **Decimal numerals:** The main purpose of this section is to introduce the idea of counting of the large numbers. The process dates back to the ancient days when people could count a large number of objects in terms of 20 or 10. They will count three times 20 and 5. Similarly, we know that if we have large numbers of 10 paise coins with us, what we do to count them? We arrange them in heaps of 10 each and each heap is given a new name, a rupee. In this system if we want to count any number of elements, we first arrange them in heaps of ten elements. Each heap is called tens and those left are called units. Now if the number of heaps of tens is larger than 10 then again we make heaps of 10 tens and we call each such heap hundreds and so on. Thus, in the given example in the book there are 2 hundreds, 5 tens and 3 units i.e. there are 2 heaps of 10 ten tens, 5 heaps of ten units, and 3 units. We express this as 253. The first element from the right expresses units, 2nd expresses 10, 3rd expresses hundred and 4th expresses thousands etc. For the absence of any units or tens etc. the symbol 0 is used there. This is how one can introduce this called the *place value system*. In the above example, place value of 3 is 3 units, 5 means 5 heaps of tens i.e. 50 units. Thus place value of 5 is 50 and place value of 2 is 2 hundred denoted by 200. The symbol 3074 represents 4 units, 7 tens, no hundreds and 3 thousands.

Though decimal system is the most convenient system of all the systems, though it is used throughout the world today, it may seem interesting to explain some other systems say, in base 2 or in base 5, to the students. This may provide a better understanding of the decimal system.

While talking about 'base five', the teacher may start with the basic symbols such as 0, 1, 2, 3, 4. Let us consider the following collections.

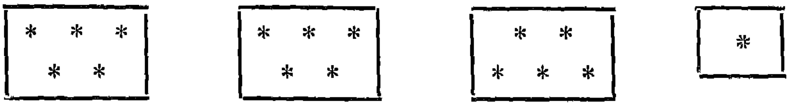


Fig. 24

In decimal numerals the number of elements above is 16. Using the symbols 0, 1, 2, 3, 4 we write $16 = (3 \times \text{five}) + (1 \times \text{one}) = 31$ five. The teacher should see it that the name thirty one is not confused with for 31 five because thirty one is 3 tens and one more. We shall simply read "three, one base five". Similar examples may be considered.

The teacher should also introduce the concept of successor carefully.

New Concepts: The following are the new concepts:

(*i*) Infinite set
(*ii*) Decimal system
(*iii*) Successor

Vocabulary:

(*i*) Infinite set

(*ii*) Decimal
(*iii*) Successor

2.4. **Number-Ray and Order Relations:** The teacher should explain to the students that in place of ray, we can also take a line. Any point on the line can be taken as the starting point and that point will be denoted by 0. The teacher may specifically mention that the distance of 1 from 0 may be taken arbitrarily, but other points must be equi-spaced, equal to the distance between 0 and 1. The teacher should show to the students that the comparison of numbers and other properties of numbers can be explained by this representation of natural numbers.

New Concept: Number line.

2.5. **Addition in W:** Ask a student to add 5 and 4. If the student has not learnt the addition table, what should he do to answer the query? He counts 5 finger tips, then he counts another 4 finger tips and then counts together all these 9 finger tips counted earlier and gives the reply. Similarly, the teacher may take 5 balls and 4 chalk pieces, combine them together and count the elements of this new set and the student will find the number of elements as 9, which is same as $5+4$.

Concept: Addition in W.

Vocabulary: Addends, addition

2.6. **Properties through patterns:** This section dealt with some structural properties of addition, e.g. commutativity, associativity, closure etc. The teacher should not introduce them by setting forth their definitions. Instead he should ask the pupil to do such sums so that he may himself discover these properties. In particular, the teacher should ask his pupil to add 4 to 5 and then add 5 to 4 and let the pupil observe that sum in both cases is same. He should ask few more such questions and verify them every time.

$$a+b=b+a$$

The teacher may ask their pupils to verify, by taking a sufficiently good number of pairs of natural numbers, that the sum of any two numbers is a natural number. We call this property the closure property and say that the set W is *closed* (or *stable*) with respect to addition.

Similarly the teacher may ask the students that if A, B, C be three disjoint sets, say A consists of 4 red balls, B consists of 5 white balls and C consists of 7 blue balls, then if we unite A and B and then the union set is added to set C or first we unite B and C and then A be added to the union set or we may unite A and C first and then unite C. The resulting set is the same set consisting of 4 red, 5 white and 7 blue balls, i.e.

$$(A \cup B) \cup C = A \cup (B \cup C) = (A \cup C) \cup B$$

which leads to the result that

$$(a+b)+c=a+(b+c)=(a+c)+b=a+b+c$$

i.e. parenthesis are not of much consequence here. The teacher may get this property verified by the students and then tell them that this property is known as associative property of addition or we say that addition is associative. To establish that the set of whole numbers is ordered, the teacher may take numerical examples and may use this property to define the successor of a whole number. A good number of numerical examples may be taken up carefully to establish that $a+x>b+x$ when $a>b$ and x is any whole number.

Concept: Order relation in the set of whole numbers.

Subconcepts: Commutativity and associativity of addition, additive identity, closure property of addition.

Vocabulary: Order relation, closure, commutative and associative properties, additive identity.

2.7. **Addition process:** The pupil has already learnt the process of addition of large numbers. In this section the different steps leading to addition may be set out by the teacher. Suppose we want to add 638 and 279.

First step: Add 8 units and 9 units, which gives 17 units which is equal to 1 ten and seven units. Thus we write 7 at the unit place and all 1 to tens, and write 1 above 3, as follows:

$$\begin{array}{r} 1 \\ 638 \\ 1279 \\ \hline 7 \end{array}$$

Second step: Now we add 1 ten, 3 tens and 7 tens and get 11 tens.

11 tens is equal to 1 hundred and 1 ten. Thus in 2nd step we write 1 at tenth place and add 1 to hundreds by writing it above 6 i.e.

$$\begin{array}{r} 1 \\ 638 \\ 279 \\ \hline 17 \end{array}$$

Third step: Now we add hundreds and we get $1+6+2=9$. Sum is less than 10, so there is no need of regrouping and we write 9 at hundred place. Thus:

$$\begin{array}{r} 638 \\ 279 \\ \hline 917 \end{array}$$

Teacher may ask more sums and ask the pupils to explain.
Vocabulary: Expanded notation.

2.8. **Substraction in W:** In earlier section we have associated addition with union of disjoint sets. Now we shall associate the subtraction to the removal of a sub-set. If we want to subtract 3 from 5, we take a set of 5 either balls and remove 3 out of it, what is left is 2 balls, and we say

$$5-3=2$$

In this process it is immaterial whether we take 5 balls, or 5 pebbles or 5 chalk pieces or 5 chairs. What is important here is that we must take a set whose cardinal no. is 5, and then we should remove the subset of it whose cardinal number is 3. Thus if $A-B$ is the set of all those elements of A which are not in B, then

$$n(A)-n(B)=n(A-B) \text{ if } B\subset A$$

Thus associating subtraction with the removal of sets, many of the facts can be easily explained to the pupil, as follows:

(i) A subset cannot be larger than the origin set. Thus $a-b$ has a meaning in W only if $b>a$. Thus 5–3, 6–4 have meaning in W, but $3-5$ or 4–6 do not have any meaning in W.

(ii) From A when we remove B we get A–B, and if we remove A–B, we get B, similarly if we unite A–B and B we get back A i.e.

A–B and B are complement of each other. This leads us to the fact that if $a-b=c$ then $a-c=b$ and $a=b+c$.

The above can be explained with the help of Venn diagrams.

(iii) Subtraction is a process which is the inverse of addition. The pupil has already been introduced to number line. The process of subtraction can be associated with that as well. If we want to add 3 to 5, we start from point 5, move 3 units on the right i.e. pass 6, 7 and reach 8. In subtraction we move towards left. Thus if we want to subtract 3 from 5 we move 3 units on left from 5 and reach 2. Thus $5-3=2$.

That the set of whole numbers is not closed for subtraction can be demonstrated by taking numerical examples. Let us take whole numbers 5 and 7. Now $7-5=2$ but $5-7=-2$ and -2 is not a whole number. Therefore, the subtraction operation on two whole numbers does not always give a whole number.

Concept: Subtraction in W.

Vocabulary: Inverse operation.

2.9. **Subtraction procedure involving large numbers:** Subtraction process of large numbers is explained with suitable examples. The teacher must bring out clearly to the pupil where it is necessary to bring 1 ten units in the given examples. Similarly the necessity of re-grouping hundreds and bringing 1 hundred to tens in example 2.

2.11. **Multiplication in W:** Multiplication of a natural numbers m by n is associated with union of m equivalent disjoint sets of cardinal number n thus:

$m \times n = n + n + n \ldots . m$ times.

Also $a \times b$ is associated with the cardinal no. of cartesian product $A \times B$ where $n(A)=a$ and $n(B)=b$, but the teacher should see if the pupils are receptive to it or not and should develop this concept only if, he finds that his pupils are receptive; otherwise it should be avoided.

Concept: Multiplication in W.

2.12. **Properties of Multiplication through patterns:** The various structural properties of multiplication should be given to the pupils, but should be developed only by means of verification of them through a large number of suitable examples and then establishing the properties by appealing to intuition alone. Thus students should verify that:

$3\times 6=6\times 3,\ 5\times 4=4\times 5,\ 7\times 8=8\times 7$ etc.

which leads to the result that

$a\times b=b\times a$ for any $a, b\in W$

i.e. multiplication is commutative in W. The teacher should then tell the students that this property is true, though the proof be avoided at this stage. But for inquisitive pupils, he may exhibit by taking m equivalent sets, arranging them one below the other and then taking equivalent sets of columns. For instance, 3 sets each of cardinal number 4 may be arranged as follows:

	Column 1	Column 2	Column 3	Column 4
Row 1	a	b	c	d
Row 2	p	q	r	s
Row 3	u	v	w	x

Their union is equivalent to 4 equivalent sets each consisting of one of the four columns. Each column set has cardinal number 3.

$4\times 3=3\times 4$

The teacher similarly should get verified the distributive property:

$a(b+c)=a.b+a.c$

and then give this property. The point that should be impressed upon the students is that this is a very useful fact about multiplication and helps solving orally many problems in everyday use. Also the teacher may indicate that even in the case of multiplication of large numbers, this is the property which is used to obtain the product.

The teacher may emphasize that the behaviour of 1 (unity) for multiplication is similar to that of 0 (zero) for addition. Thus, 1 is called the identity for multiplication.

Vocabulary: Multiplication table, commutative property of multiplication, associative properties of multiplication, multiplicative identity, factor, product, tree diagram, tabular form.

2.14. **Division in W:** In the earlier section, the multiplication is associated with union of disjoint and equivalent sets. Now we can associate similarly, the division to the breaking up of a set into equivalent and disjoint sets. Thus if, we have a set of 8 elements, we can break it up into 4 sets each of two elements, thus we say that

$$8 \div 2 = 4$$

Whenever we unite any number of equivalent sets, we always get a finite set, i.e. two numbers can always be multiplied. But it may not always be possible to partition a set into disjoint subsets having the same given cardinal number. Thus a set of 10 elements cannot be broken up into disjoint equivalent sets of 3 elements. The question that arises is that when is it possible? Naturally, we observe that if the set A is obtained by uniting sets equivalent to B, then it can be partitioned. Thus, we are led to the fact that a is divisible by b only if a is some multiple of b, i.e. if there exists some c such that $a = bc$. This c is called the quotient $a \div b$. Since $a = a.1$ if $a \neq 0$. Therefore, we can say that every natural number is divisible by a and itself. Since $a.0 = 0$.
Therefore, no set can be partitioned into finite sets of sets equivalent to empty sets. Thus we have an important fact.

Division by zero is not admissible

Is 10 divisible by 5? Yes, since $10 = 5 \times 2$ and $10 \div 5 = 2$.
Is 9 divisible by 4? No, since $4 \times 1 = 4$, $4 \times 2 = 8$, $4 \times 3 = 12$.
Thus there exists no number m such that

$$4 \times m = 9$$

But if we want to break up a set of 9 elements into subsets of 4 each, we get two subsets of 4 elements and 1 such set is left with 1 element. This is stated as

$$9 = 2 \times 4 + 1$$

It should be pointed out to the students that when we perform the operations (such as addition, multiplication, division etc.) with a single digit, we do it from the elementary tables. But when we need operations for computations involving multi-digit numerals, we apply the procedure what is called an algorithm.

The technique of obtaining values of r and s from $a = rb + s$, $0 \leq s < b$ is called the division process explained below, by means of an example, where $a = 39146$ and $b = 19$.

$$\begin{array}{rcccccc}
 & & & 2 & 0 & 6 & 0 \\ \cline{3-7}
19) & & 3 & 9 & 1 & 4 & 6 \\
 & & 3 & 8 & & & \\ \cline{3-4}
 & & & 1 & 1 & & \\
 & & & 0 & 0 & & \\ \cline{4-5}
 & & & 1 & 1 & 4 & \\
 & & & 1 & 1 & 4 & \\ \cline{4-6}
 & & & & & & 6 \\
 & & & & & & 0 \\ \cline{7-7}
 & & & & & & .6
\end{array}$$

Explanation: The dividend consists of 3 ten thousands, 9 thousands, 1 hundred, 4 ten and 6 units; we take

Ist step: Ten thousand as a unit, 3 such units divided by 19, gives 0 as quotient and the remainder is the dividend 3 itself; hence there are no tenth thousands in the quotient; strictly speaking, we have to write in the beginning of the quotient 0 i.e. 02060 which is not necessary in the notation employed by us.

2nd step: Hence we take 39 thousands and taking thousand as a unit divide by 19, we find that the highest multiple of 19 which does not exceed 39 is $19 \times 2 = 38$; we write this as 38 in the second row, subtract and get 1 as remainder which is shown as the first digit in the third row. We shall explain the symbols employed here. We are writing 2 in the quotient and 38 in the second row; both these are the corresponding thousands; the corresponding zeros are omitted for the sake of simplicity in writing.

3rd step: The first digit 1 in the third row represents 1 thousand (because we divided 39 thousands and obtained it as remainder). We take 1 (the next digit of the dividend) which is really 1 hundred and write it as the second digit in the 3rd row; and we consider it as 11 hundreds and taking now *hundred* as a unit, divide 11 by 19. The highest multiple of 19 which does not exceed 11 is 0, hence we write 0 as the second digit in the quotient, which represents 0 hundreds and write the multiple of 19 by 0, as 00 below 11 and get the remainder 11 which is 11 hundreds.

4th step: We consider 11 now not as 11 hundreds but as 110 tens and add to it 4 tens (the 4th digit in the quotient) and write this as 114 in the 5th row, which represents 114 tens and carry out the multiplication by 19, taking tens as units. The highest multiple of 19 which does not exceed 114 is its 6th multiple which is 114 itself; we write 6 in the quotient as the third digit which naturally represents 6 tens, the remainder being 0 which we omit for simplicity and take which is 6 units now.

5th step: We have to divide the 6 units or 6 merely by 19; the highest multiple of 19 which does not exceed 6 is its 0 multiple, hence we write 0 in the 4th place in the quotient which represents 0 units and write the 0-multiple of 19 i.e. 0 below 6 and get the remainder as 6.

Concept: Division in W.

2.16. **Exponents:** Experiments are introduced here in a situation which shows clearly their usefulness for concise notation. The students' attention may be drawn to the similarity of patterns for repeated addition and repeated multiplication.

Concept: Exponent.

Vocabulary: Exponents, power, raised to the power base.

2.17. **Factors and Multiples:** It should be made clear to the students that zero is not a factor of any number, but a multiple of any number.

Concepts: Factors and multiples.

Vocabulary Factor, multiple, divisor, common factors, common multiples, improper factors, proper factors.

2.18. **Even and Odd Numbers:** Zero being multiple of any number, it is a multiple of 2 also. Hence we take zero in the set of even numbers.

Concepts: Even and odd numbers.

Vocabulary: Even, odd.

2.19. **Prime and Composite Numbers:** The special features of prime numbers ought to be emphasized. It is to be noted that all even numbers are composite numbers; use of exponents in complete factorisation should be encouraged.

Concepts: Prime and composite numbers.

Subconcepts: Relatively prime.

Vocabulary: Prime numbers, composite numbers, resolution, prime factorisation, trivial, Sieve of Erastosthenes twin primes, relative primes.

2.20. **L.C.M. and H.C.F. of Natural Numbers:** Instead of abbreviations L.C M. and H.C.F, the complete phrases 'Least common multiple 'Highest common factor' should be used so that the students could understand its property. A property common to several things may be made clear to the students by taking different examples from the surroundings. Zero being a multiple of all number, it is the least common multiple. However, by least common multiple we mean the smallest counting number in the set of common multiples Before introducing the shortcut method, the students may be given a good number of examples of finding H.C.F. of two numbers and the each of the number and their difference.

Concepts: H.C.F., L.C.M.

Vocabulary: Highest common factor, Least common multiple, Euclidean Algorithm.

3. Understanding and skills to be achieved

It hardly needs any reiteration that our approach to the teaching of mathematics at this level leans heavily on what is sometimes called 'discovery approach' and accordingly, this approach has to be oriented in such a way that it would enable the students to understand the key ideas that this chapter deals with. Thus, briefly, the ideas to be understood and grasped thoroughly by the students are the following:

(*i*) one-to-one correspondence
(*ii*) equivalent sets (finite)
(*iii*) the concept of larger or smaller sets
(*iv*) cardinal numbers
(*v*) ordinal numbers
(*vi*) counting (natural) numbers
(*vii*) concept of zero
(*viii*) decimal numerals
(*ix*) place-value systems
(*x*) the number line
(*xi*) the concepts of 'greater' and 'less'
(*xii*) basic operations on natural numbers together with their

properties.

(*xiii*) number patterns and relationships

Let us now look into these ideas in more details. It is necessary that the notion of one-to-one correspondence has to be introduced through comparison or matching of the sets, which will eventually lead him to the idea of equivalent sets. The distinction between equivalent sets and equal sets may be stressed. It may not be out of place to mention here the studies made by psychologists such as Piaget, Dienes and others on the formation of these mathematical concepts. To Piaget 'the formation of a concept takes far longer than had been supposed' and as such, the slant of teaching has to be more on creation of 'learning situations' than on anything else. The teacher should be alert on the responses of the students when they are involved in experiments on one-to-one correspondence. Since language has a vital part in the learning of mathematics, the teachers should see that the students learn the appropriate mathematical language, 'larger than', 'less than' etc. As Dr. Lunjer, in his article *The Work of Piaget and its Relevance for the Teacher*, has remarked, 'number is only one of the instruments whereby we carry out the ordering and systematisation of experience'; the teacher should necessarily be watchful how the students grasp the concept of a number. It is essential that the students should acquire what Piaget has called 'conservation' in number, which means that the number of objects in a group remains the same, however, the objects are arranged. The next topic is about counting the number of objects *vis a vis* a cardinal number. About cardinal numbers the students should know that every set equivalent to { 1 } has the cardinal number 1, every set equivalent to { 1, 2 }, the cardinal number 2 etc. That the concept of cardinal numbers is associated with total number of elements in the set, while the ordinal number is associated to a particular position of the element should be clearly explained. The number line and the numbers in sequence should form the next topics which the students are required to understand and the teachers should be watchful that these ideas begin to develop in a certain awareness of place-value in the number notation and finally, the full-significance of place-value should be explained to the students. The comparison of numbers on a number line should be explained to the students. The teacher should explain that on the number line, any natural number can be taken as a starting point. While teaching decimal system, its advantages ought to be emphasized along the following lines.

(1) In this system, only ten basic symbols (which we call digits) viz. 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are needed to write a number and no additional symbols are necessary to introduce any larger number.

(2) Place value in which each position corresponds to a power of the base is of importance in a system used for calculations.

(3) The concept of zero is essential to the development of the place-value system.

The difference between a number and a numeral should be clearly brought out. The students should be told that number is an abstract idea and numeral is the means of expressing that idea through symbols. Incidentally, the teacher may explain the difference between a numeral and a digit as outlined in the textbook and in the general notes of this chapter. While dealing with decimal system, the teacher should point out clearly that in it only 10 digits are used and the emphasis here is on place-value. The number is changed by changing the place of a digit in a numeral i.e. 635 as different from 653. The use of place-value in the structure of the decimal system i.e. units tens, thousands etc. should be grasped by the students. That a decimal system requires, only a few symbols to write large or small numbers, that in a decimal system, the position of each digit corresponds to a power of the base and that zero is a place holder should be explained thoroughly to the students. The system in base 2 which we call a binary system has been very conveniently used in the high speed computing machinery.

The teacher should see that the students understand the basic operations with the help of the number line and it should be emphasised that these are properties or the operations and not dependent on the enumeration system.

A teacher should make his students aware of the order and pattern in mathematics and they are able to recognise them wherever they may decipher. The students will be able to acquire the skill in arriving at generalisations, if they understand the terms 'factor', 'primes', 'common factors', 'common multiples' etc. The students should be in a position to find out the factors of a composite number, to work out the H.C.F. and L C M. of numbers.

Although all the foregoing topics will help the students to discover something of the underlying structure of the system of whole numbers, the teacher should not lose sight of the necessity of computational aspect of the studies. The chief purpose of computation lies in the written recording of various operations and relationships. These, obviously, require a good deal of practice by students without which, according to another eminent psychologist Dr. Dienes, it may not be possible to fix up a concept in the minds of the students.

4 Class room and student activities

It goes without saying that the environment or what often goes by the name 'learning situations' is a necessary adjunct in the teaching of mathematics at this level.

The first topic to be introduced is about one-to-one correspondence. We may refer now to one of the well-known experiments carried out by Piaget in the course of his researches on the formation of concept in the mind of young students. And an interested teacher might profit by this example of Piaget. We would now refer to a similar experiment that the teacher might conduct in the class. The teacher should ask a student of the class to tell him whether the number of students present in the class is the same as the number of students' chairs in the class or less. The students may be a bit puzzled by this question. The teacher will then point out that each boy is sitting on a chair and there is no vacant chair, i.e. every chair has a student on it. To every chair, the teacher will say, we can associate or match a student and vice-versa. The teacher will then say that this is what is known as one-to-one correspondence between the sets. The equivalence between the set of students and set of the chairs should be brought home to the students.

It would be an interesting class-room activity to impress upon the students, the need for the concept of cardinal numbers. The teacher may pose the question, how many brothers and sisters do you have? How many books are there with you have to read at home? Initially, the students may not be able to give answers to these questions. The teacher may remove the difficulty conveniently by citing examples etc. to establish always a equivalence between sets and thus, arises the need to form a set of abstract symbols and to give each set some name. These will facilitate the introduction of one-to-one correspondence and also equivalent sets leading to the idea of some cardinal numbers. The teacher may, now, ask a student to count the number of alphabets

A, B, C, D, E, F, G

which are to be writen on the board. The teacher should help the students to establish

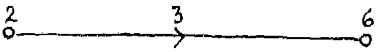
$$\begin{array}{ccccccc} A & B & C & D & E & F & G \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

and so the students would come to the conclusion that counting is but establishing of equivalence to a subset { 1, 2 } The teacher nothing may tell them that the cardial number of the empty set is denoted by the symbol 0 read as 'zero' or 'shunya'.

The use of a structural apparatus, a flow diagram or a machine will go a long way in getting across the students the basic operations. Of course, the important class room activity in respect of these operations consists in oral and computational practice. A few real problems would be of much interest to the students.

As regards 'number patterns', 'prime numbers' etc., some work books may be prepared by the teachers for setting interesting tasks in the class as well as at home. As a part of this activity, the Sieve of Eratosthenes for numbers upto 100 may be set for construction by the students.

The students may be asked to make diagrams to represent complete factorisation of a number. Basically, we can represent a product by the diagram



, where 3 associated with the line is the multiplier, which "takes 2 into 6". The arrow indicates the direction in which the multiplication goes.

The different complete factorisation of the number 18 may be represented by the following diagram.

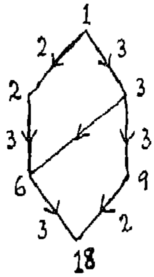


Fig. 25

The figure shows that the three complete factorisation of the number 18 are $1\times2\times3\times3$, $1\times3\times3\times2$, $1\times3\times2\times3$

Drawing such figures we can find the H.C.F. of numbers

5 List of concepts, sub-concepts, vocabulary, symbols

Concepts : One-to-one correspondence, Equivalent set, Cardinal number, Counting numbers or natural numbers, Ordinal numbers, Set of whole numbers (W), Infinite set, Decimal system, Number line, Successor, Addition in the set of whole numbers, Order relation in the set of whole numbers, Subtraction in the set of whole numbers, Division in the set of whole numbers, Exponent, Factors and multiples, Even and odd numbers, H.C.F., L.C.M., Prime and composite numbers

Sub-concepts: Commutativity and associativity of addition, Additive identity, Closure property of addition, Relatively prime

Vocabulary: Cardinal number, Natural number, Ordinal number, Infinite set, Decimal, Successor, Addends, Order relation, Closure, Commutative and Associative properties of addition, onto mapping, into mapping, Additive identity, Expanded notation, Inverse operation, Multiplication table, Closure, Commutative, Associative properties of multiplication, Multiplicative identity, Factors, Product Tree diagram, Tabular form, Even, Odd, Prime numbers, Composite numbers, Resolution, Prime factorisation trivial, Relatively prime, Sieve of Erastosthemes, Euclidean Alogrithm.

CHAPTER III

THE SET OF FRACTIONAL NUMBERS

1. Aims and objectives of the chapter:

The present chapter is, in essence, a continuation of the previous chapter in which we have dwelt at length on the system of whole numbers. This chapter introduces to the students, another number system, known as positive fractional numbers. As in the earlier chapters, we will see here that there are certain properties which are held in common by all these number systems and one of the most important objectives of this chapters is to emphasize these *operations*, as we often put it. Of course, there are certain properties which hold in some number systems but not in others and these, therefore, enable us to distinguish one system from the other. Although the students in this chapter will be acquainted with ideas such as fractions and decimals, it is felt that this chapter will provide the students with situations that would help them to see what is sometimes called "the skeleton of a mature mathematical system" without being smothered by the more subtle ideas which are necessary for complete understanding.

2. Method of introducing concepts:

General notes: Historically, the invention of fractions was associated with problems of sharing and it may be mentioned that, the famous Rhind Papyrus (1700 B.C.) had a set of tables showing how to express fractions in terms of unit fractions and this was done earlier than that of introducing negatives. The Babylonians used fractions; the Romans were also acquainted with fractions and the Greeks thought of fractions (rationals) in geometrical terms. We describe a fraction by a pair of numbers: a denominator, which gives the size of the denomination we are measuring and a numerator, which enumerates how many units of this denomination there are. This pair of numbers is an ordered pair, for one must be aware in a fraction, which of the two numbers is a numerator and which, the denominator. The convention of denoting this is to write the numerator above the denominator with a bar between them which, it is believed, came into vogue, in the sixteenth century. We might, likewise, define a rational number as the ordered pair of numbers and there is a certain defect in this. For example, three-quarters gives the same share as six-eighths or as nine-twelveths. In otherwords, there are infinitely many fractions that are '*equivalent to*' three quarters and we denote this *equivalence* by the symbol $\cong$ and we write:

$$\frac{3}{4} \cong \frac{6}{8} \cong \frac{9}{12} \cong \quad \ldots\ldots\ldots\ldots\ldots$$

This leads us to the definition of equivalence of fractions in terms of rational numbers (to begin with). We say that $\frac{a}{b} \cong \frac{c}{d}$, if and only if the natural number a.d and b.c. are equal. It is easy to verify that the relation $\cong$ is an equivalence relation, for

(*i*) $\frac{a}{b} \cong \frac{a}{b}$

(*ii*) $\frac{a}{b} \cong \frac{c}{d} \Rightarrow \frac{c}{d} \cong \frac{a}{b}$

(*iii*) $\frac{a}{b} \cong \frac{c}{d}$ and $\frac{c}{d} \cong \frac{e}{f} \Rightarrow \frac{a}{b} \cong \frac{e}{f}$

The relation $\cong$, therefore, partitions the fractions (i.e. the ordered pairs) into equivalence classes, for example:

$\left[\frac{3}{4}, \frac{6}{8}, \frac{9}{12}, \frac{12}{16}, \frac{15}{20}, \quad \ldots\ldots\ldots\ldots\ldots\right]$ is an equivalence class. We call this complete equivalence class a *rational number* or strictly speaking, an *unsigned rational number*. The set of positive rational numbers is denoted by+. Our next step would be to bring the system of rational numbers within the framework of number systems and for this purpose, we have to define the sum and product of two rational numbers.

Let there be two equivalence classes containing the fractions $\frac{a}{c}$ and $\frac{b}{c}$. The equivalence class which contains the fraction $\frac{a+b}{c}$ is called the sum of two rational numbers. Similarly, the equivalence class containing the fraction $\frac{ac}{bd}$ is the product of the fractions contained in the equivalence classes $\frac{a}{b}$ and $\frac{c}{d}$.

If one rational number is divided by a non-zero rational number, we get a rational number For, let u, v and w ($v \neq o$) be three rational numbers such that $u=vw$, then we write $w=u \div v$ or we say that v divides u. The property of transitivity also holds for the set of rational numbers i.e. $u > v$ and $v > w \Rightarrow u > w$ where u, v and w are all rational numbers. The monotony laws are true for this order relation in the set of rational numbers. That is, $a > b \Rightarrow a+x > b+x$ where a, b, x are all rational numbers.

The rational numbers can be represented on a line and it can be shown, as in the previous chapter, and as dealt with in the book and from this representation, some important properties of the set of rational numbers Q emerge and these are as follows:

(1) Q is *ordered.*

This means if there be rationals x, y, z such that $x<y$ and $y<z$ then $x<z$. In this case, we also say y is in between x and z.

(2) Q is *dense.*

That is, in between two different rational numbers there lie an infinite number of rational numbers.

(3) Q is *Archimedian.**

That is, the four operations satisfy the following laws:

(1) Commutative laws of addition and multiplication,
(2) Associative laws of addition and multiplication,
(3) Monotony laws for addition and multiplication i.e. if $a<b$ then $a+c<b+c$ and if $a>b$ and $c>o$ then $ac>bc$,
(4) $(a+b)\ c=ac+bc$ (i.e. right distributive law).
(5) $b.\frac{a}{b} = a$ ($b \neq o$) (law for division).

If we are to extend the place-value system to the non-integral part of a rational number, so that it will facilitate our computations with rationals, we have to introduce what are called 'decimals'. A formal definition is as follows:

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*That is if a and b are any two positive rational numbers, an integer exists such that $nb > a$.

Definition:

The decimal $\pm A\cdot a_1 \; a_2 \; a_3 \; \ldots . a_n \; . \; \ldots .$ in base b is the rational number.

$$\pm \left(A + \frac{a_1}{b} + \frac{a_2}{b^2} + \frac{a_3}{b^3} + \ldots\ldots\ldots\ldots\ldots + \frac{a_n}{b^n}\right)$$

Thus in base 2, the decimal 0.1 is $\frac{1}{2}$, .001 is $\frac{1}{8}$ and so in general in base b the value of the nth place after the point is $\frac{a_n}{b^n}$.

Any rational number may be expressed as a decimal expression if we choose the base. But when the base is given beforehand, then sometimes difficulties arise. For example, $\frac{1}{3}$ in base 10 is .3333...... It is a non-terminating decimal, it is periodic and can be written as $.\dot{3}$. $\frac{1}{3}$ is the limit or the sequence of numbers

0.3, 0.33, 0.333,. ..... etc.

It may be shown that all rational numbers may be expressed as exact decimals or as the limits of repeating decimals. In general, a rational number $\frac{a}{b}$ is said to be a decimal fraction, if it is equal to a fraction whose denominator is some power of 10. Next, the operations on decimal fractions. The sum and product of two decimal fractions is either a decimal fraction or a whole number.

For, let $\frac{a}{b}$ and $\frac{c}{d}$ be two decimal fractions and they are in their lowest terms.

Then $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

Now 2 and 5 are the only two prime factors of both b and d. So the product bd has also no prime factor other than 2 and 5. So if $ad+bc<bd$, then $\frac{ad+bc}{bd}$ is a decimal fraction and if $(ad+bc)$ is a multiple of bd then $\frac{ad+bc}{bd}$ is a whole number.

Again $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

if ac is a multiple of bd then $\frac{ac}{bd}$ is a whole number and if $ac<bd$

then $\frac{ac}{bd}$ is a decimal fraction. But in case of division of two decimal fractions, we may not get a decimal fraction.

For let $\frac{a}{b}$ and $\frac{c}{d}$ be two decimal fractions, so that b and d have the two prime factors 2 and 5.

Now $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$

but may not have 2 and 5 as two prime factors; it may have other prime facotrs. So we may conclude that $\frac{a}{b} \div \frac{c}{d}$ may not be a decimal fraction but a rational number.

3.1 & 3.2 *Introduction & Equivalent Fractions*:

In order to introduce positive rational numbers, the teacher may follow the historical method adopted by Egyptian; that is, by taking two mugs, one big and another small. Then, the teacher will pour water from the small mug to the big mug and say that the mug is filled up after pouring water 7 times by the small mug. Then the teacher may ask the students, 'What is the relation between the small mug with the big mug?' It is expected that few intelligent students will answer that the small mug is one seventh of the big mug. Then the teacher will write 1/7 on the board. He (or she) will explain to the students that now we have got a number which is not a counting number but a counting number divided by a whole number.

Now, to explain the equivalent fractions, the teacher may take a piece of rectangular sheet of paper and then, he will fold the paper in such a manner that the sheet is being halved. Then, the line of folding divides the sheet of paper into two equal halves.

Again, if the teacher takes the (equal) same sheet of paper and divide it into four equal parts, then each part is equal to 1/4 and also 2 parts is equal is 2/4 and so, it is clear that 2/4 and 1/2 are the same. So we can write $2/4 = 1/2$.

The reason behind it is that if a, b and k are whole numbers and b and k not zero then $bx = a$ i.e. $x = \frac{a}{b}$ is a positive rational number. Since bx and a are names for the same number, k(bx), (kb)x and ka are different names for one number.

If (kb)x = ka or $k = \frac{ka}{kb}$ we can say $\frac{a}{b}$, $\frac{ka}{kb}$ are equivalent fractions. The teacher will explain that the simplest fraction for a rational number is a fraction in which the numerator and denominator have no common factors except unity.

Concept: Fraction.

Vocabulary: Denominator, numerator, fractions, equivalent fractions.

3.3 *Order*:. The idea of fraction (positive rational number) may conveniently be introduced by using a number line. The students have already become acquainted with a number line in the previous chapter. Now, in this chapter, they may be shown that between 0 and 1 on the number line, there are many other points and the distance of all these points from 0 are leas than 1. Then, if any one takes a point just in the middle of 0 and 1, the teacher will be asked to represent $\frac{1}{2}$ at the middle point on the line. The students should write half and this half can be written as 1/2 on the board.

Now one function is greater than the other fraction, if two fractions have the same denominator but different numerators; the fraction with the greater numerator is greater than the other fraction. Now, if two fractions have the same numerator but different denominators, then the fraction with lesser denominator is the greater fraction.

Concept: Order: 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11: addition in Q_+, properties of addition in Q_+ etc.

When the product of two rational numbers is 1, those numbers are called reciprocals of each other. The students know that $\frac{k}{k}$ is 1 from their basic knowledge of division of natural numbers.

As $k. \frac{1}{k} = 1$, we can say that when 1 is divided by k we get $\frac{1}{k}$ and when 1 is divided by $\frac{1}{k}$ we get k; so each k and $\frac{1}{k}$ are reciprocals to each other. By using number lines, the teacher may explain improper fractions i.e. when the fraction is equal to some whole number plus a proper fraction. Here, it is to be mentioned that a proper fraction is a fraction (the rational number) where the numerator is less than the denominators. As in the case of natural numbers, the teacher should emphasize that the rational numbers obey the commutative and associative laws and distributive law.

In order to introduce the concept of multiplication of rational numbers, say, $\frac{1}{2} \times \frac{3}{4}$, the teacher may take a rectangular sheet of paper and then divide the sheet into 4 equal parts, one part is equal to 1/4.

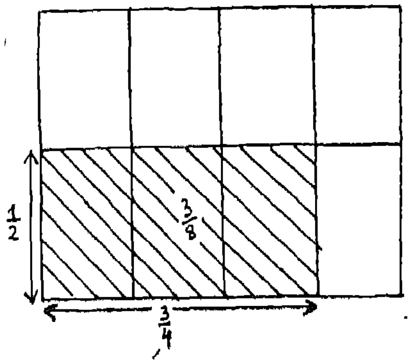

Fig. 26

Now, he will take 3 parts, so 3/4 is taken. By halving, we get 3/8, because each 1/4 is being halved gives 1/8.

The concept of division of rational number may be made use of to clear the idea that $\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$ (say) when 3/8 is divided by 1/2, we get 3/4 or when 3/8 is divided by 3/4, we get 1/2.

In order to introduce the concept of addition of positive rational numbers, say, 3/4+1/2, we first take a sheet of paper and divide the sheet into 4 equal parts then we get 3/4 of the paper as 3 parts of the sheet. Then 2 parts of the sheet is equal to 1/2 as 2/4=1/2. Therefore to add 3/4 with 1/2 is the same as to add, 3 parts with 2 parts that is 5 parts which is equal to 5/4 of the whole sheet. Hence, we can write

$$\frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

The idea of exponent can be explained as in the case of natural numbers.

Concepts: Addition, subtraction, multiplication and division of fractions, exponents.
Subconcepts: Additive identity, multiplicative identity, reciprocal, additive inverse, multiplicative inverse.
Vocabulary: Identity, Inverse—additive and multiplicative.

3.12 *Ratio*: a:b is the same as the positive rational fraction. When two ratios are equal we say that the two ratios are proportional. Let a:b and c.d be two ratios.

Then a·b :: c·d is the same as $\frac{a}{b} = \frac{c}{d}$

Concept: Fractions as ratios.
Vocabulary: Ratios, fractions.

3.12, 3.13, 3.14, 3.15, 3.16:*Fractions as Ratios, Fractional Numbers*:

The teacher may peel off an orange and show that there are 10 pieces of orange. The teacher may now pose the question, 'What is the relation between one piece with the whole unbroken orange?' The answer is quite simple, 'it is 1/10 and now the teacher may write 1/10 on the board and will explain that this 1/10 can also be written in another form i.e. 1/10=.1.

The teacher may then ask the question, 'What is the relation between one paise and a rupee?' The answer expected to be is 1/100. The teacher may proceed to explain that $1/100=1/10^2$ and it may be written as .01. Hence the concept of decimal places may be explained by chart like the following:

Hundredth's place	Tenth's place	Unit's place	First decimal place	2nd decimal place	3rd decimal place
10	10	1	.1	.01	.001

Now the teacher may put the question "How can you express $\frac{123}{1000}$ in decimal?" The answer ought to be 1 in the 1st decimal place, 2 in the second decimal place and 3 in the third decimal place, so that the student will write .123 for $\frac{123}{1000}$.

The rules for addition and subtraction of decimals are similar to the rules for addition and subtraction of whole numbers. Now, how to multiply two decimal fractions, say $.8 \times .12$. This is the same as multiplying 8 by 12 and then putting the decimal point before three digits. So here, $.8 \times .12 = .096$

In case of division, the same method is applied. Say, to divide .84 by .12.

This is the same as to divide 84 by 12

For, $\frac{.84}{.12} = \frac{.84 \times 100}{.12 \times 100} = \frac{84}{12} = 7$

So when we divide a decimal fraction by a decimal fraction, say, .84 by .12 we shift the decimal points of both the divident and the divisor, so that the divisor is a whole number and then divide as in the case of natural numbers and put the decimal point where necessary.

Concepts: Decimals

Subconcepts: Addition, subtraction, multiplication and division of decimals.

Vocabulary: Decimal points.

3. Understanding and skills to be achieved

Our primary goal here, doubtless, is to create opportunities for students to discover the structures and obviously, the classroom situations do play a vital part in the understanding of some concepts and the structures involved. To begin with, the students should be encouraged to explore situations where they will have an opportunity to find that in the case of accurate drawings, the ratios of corresponding lengths can be expressed in fractional forms, for this gives rise to the word 'rational'. It is, therefore, a necessity that the students should understand the meaning of fraction and the equality of fractions. It needs to be pointed out that a fraction is a name for a number and that a number has many names. A wide variety of experiences of paper folding and cutting, of sharing a quantity or partitioning a set help to establish the equivalence of fractions, which is of utmost use in the computation of fractions.

Ultimately, the students should come to grasp the fact that $\frac{a}{b}$ represents the number x such that $bx = a$, that is, a rational number is the quotient of a whole number and a counting number. At this stage, the students have very little idea about fractions, so they cannot think of division as an inverse operation to multiplication. It should be explained that the rational number $\frac{a}{b}$ is a number which when multiplied by 'b' gives 'a' as the product, i.e $\frac{a}{b}$ is the number x for which $bx = a$. Pupils should be advised to change improper fractions into mixed numbers, but it has to be made clear to them that both the forms are correct and both of them have their uses. In this section, it is hoped that the pupils should grasp thoroughly the operations on fractional numbers. It is only after representation of fractional numbers on a number line, the children should realise that any fractional number can be divided by another number to yield a fractional number.

Moreover, the students should be able to compare two fractions and thus, through examples, they would understand that, if two fractions have equal numerators, the fraction with the smaller denominator represents the larger number. In brief, the students have to master the techniques of working with fractions. In this chapter, the students should learn that the points between any two points representing two successive numbers on the number line would represent fractions. In decimals, it should be clarified to the students that we go to the right of units while in natural numbers, we start with units, go to tens, to hundreds,...... to the left. Thus, the students will properly understand the decimals, if the teacher can recall the place-values of digits. It should be specifically understood that digits to the right of unit's digit stands for tenth part of a unit and next is for hundredth part of a unit. The students will come to realise the significance of the symbol (.) called decimal point, only when they have a feeling for a symbol which separates these parts from the whole numbers, which are called integral parts and the other fractional parts. The teacher should emphasize that values of digits in a decimal become 10 times as we move on the number line from left to right and the converse may also be pointed out The students should remember that decimal means a fraction having 10 or any power of 10 as denominator and it admits of changes into single fractions and vice-versa. There should be intensive as well as extensive drills on the decimals and their working rules.

4 Class-room and student activities

It is a cardinal point in the scheme of "active learning" that the emphasis is shifted from hackneyed instruction by teachers to investigation and discovery by themselves. But, here, again the teacher has to provide the students with situations and opportunities so that they might be in a position to invent and discover the properties by themselves. A teacher has to be watchful over the topics that will arise in the class-room. It is necessary, as in other topics, for a teacher to see that a young student learns more readily, when a student is engrossed in an activity, that student enjoys patterns that he (or she) might discover, and that a student is able to communicate his (or her) findings to his (or her) teacher and companions. A necessary concommitant to make these things a reality will be to provide the students with real materials such as boxes, leaves, flowers, cubes, beads, balls, papers, scissors etc. The teacher will select the materials that have mathematical potentials and observe its use by the students, and the teacher might then ask questions to help the students to think.

So far, the students have already been acquainted with whole numbers and they have, later learnt how to combine them according to these operations. Also, most of the students have some experience even before they come to school and the students should be given some real materials, models or cutouts (say, papers) to cut them into two or three equal shares. In the class-room, these will give to the students an experience of fractions and the use of the relevant language such as 'cut it in half', 'fold it in four' (or quarters) etc. For example, each student may have a sheet of paper and then, he, may divide the sheet into five equal parts and then, each part is one fifth of the whole sheet. So what about three parts? Now, the students can easily tell that 3 parts is equal to 3/5 of the full sheet. The student may be given some small measured container and a big container and the student may pour liquids from the measured container into the big unmeasured container and thereby, the idea of fraction can be made clear to them. A teacher may create a problem situation in the class by asking some of the students to prepare 1 cm. cubes from wood, cardboards and others, 2 cms. ones etc. Further, the weights and measures afford some experiences in the use of simple fractions. Perhaps, a number of a variety of regular geometric shapes may be used for giving the students with opportunities for recognising the common fractions. The measurements will also give to the students many opportunities to work with fractions and also, with the use of four operations applied to fractions. A few historical anecdotes relating to fractions used by the ancient people, namely, the Greeks and Egyptians, may help to stimulate curiosity among the students.

Finally, a few words about the appropriate drills to be done in the class. It may be held that the drills may be undertaken along with the work carried out by the students with the actual materials. A few examples may be set to the class on 'pure' fractions such as $\frac{3}{4} \times 120$, one half of $6\frac{1}{2}$ cm. and these may also be taken up along with the actual materials. Likewise, the teachers may give drills on 'applied' fractions, the examples of which may largely be drawn from the surroundings of the class, e.g. if in a class of 35 students 2/5 of the students were boys, how many were girls? Another sophisticated machine, which the teacher may show to the students is the stretching and shrinking machines to explain fractions, vide, Braunfeld, Dilley and Rucker (A new UICSM Approach to Fractions for the Junior High School) and Braunfeld and Wolfe (Fractions for low achievers). When we say 2/3rd of 6 cms. is 4 cms., it is supposed that 2/3 is an operator which transforms a length of 6 cms. into a length of 4 cms. In order to give a concrete shape to this abstract mathematical idea, one can think of these operators as machines which can change the length of sticks. First of all, we include the 1-machine and it does not stretch sticks, that is, this machine leaves all lengths unchanged. That is, this 1-machine may be conceived of as an identity operator. In the stretching machines, there are three basic elements, (i) an input stick, (ii) a machine and (iii) an output stick. A 2-machine doubles the length of any input stick, a 3-machine stretches each input stick to an output stick three times as long and in general, an n-machine stretches an input to an output n times as long. One may build a hook-up of a 3-machine followed by a 4-machine, so that the new output stick becomes twelve times as the input stick. We may now introduce shrinking machines. An example of a shrinking machine is the 2-machine which shrinks any input stick to one-half its original length. So, we see that the idea of rational numbers and the idea of operations (like multiplication and division) can be clarified to the students with the help of stretching and shrinking machines.

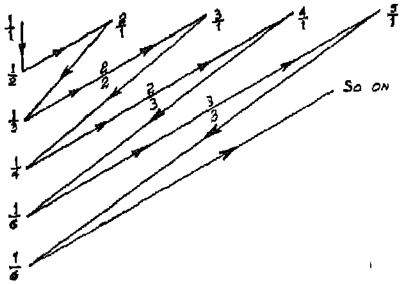

Fig. 27

The + ve rational numbers may be put into one-one correspondence with the set of natural numbers. The above figure shows that the + ve rationals may be put into an order and counted.

As regards decimals, we know that the rational number 1/2 may be written as a decimal 0.5. The teacher may draw the attention of the students of the use of decimals in real situations e.g. the milometer of a car or motor cycle etc. The students may be asked what is meant by 'one tenth' of a matter in problems of journeys. There is another method of motivating the idea of decimals and this may be done by hand calculating machines and the teacher may use them if the teacher so desires. Through many examples, the students can verify that the fractions $\frac{a}{b}$ whose b is other than 2 or 5 is a non-terminating decimal. That a rational fraction $\frac{a}{b}$ in lowest terms has a terminating decimal expansion if and only if the integer b has no prime factors other than 2 or 5 can be verified by the students through a number of examples.

5. List of concepts, sub-concepts, vocabulary, symbols

Concepts: Fractions, order.

Subconcepts: Addition, subtraction, multiplication and division of rational numbers, exponents, fractions as ratios, decimals, addition, subtraction, multiplication and division of decimals. Additive identity, multiplicative identity, reciprocal, additive inverse, multiplicative inverse.

Vocabulary: Denominator, numerator, fraction, equivalent fraction, identity, inverse-additive and multiplicative, ratios, fractions, decimal points,

Symbols:

(1) $\cong$ (Equivalence)
(2) $\Rightarrow$ (Implies)
(3) $\left\{\frac{a}{b}\right\}$ (Family of fractions)
(4) $>$ (Greater than)
(5) $<$ (Less than)
(6) 0. (Decimal point)
(7) : (Is to (ratio))
(8) : : (Proportionality)
(9) $a \rightarrow ((a, 1))$ a is mapped onto (a, 1)
(10) $a \leftrightarrow (a, 1)$ (a is isomorphic to (a, 1))
(11) $\neq$ (not equal to)

CHAPTER IV

SENTENCES

I. Aims and Objectives of the Chapter:

The primary objective of this chapter is to introduce some elementary ideas of calculus of propositions such as, 'sentences'. The operations, hitherto defined for numbers, have been used in this chapter to find the solution of some simple equations.

II. Methods of introducing Concepts

General Notes: This chapter uses some elementary ideas from the calculus of propositions and so it seems desirable that the teacher should be familiar with its essentials. It is with this idea that we sketch below, a brief outline of the elements of the classical calculus of propositions.

Propositional Calculus

Our everyday language is often vague and sometimes even ambiguous, and thus, lacks the clarity and precision essential for any deductive reasoning in any branch of systematised knowledge such as mathematics. This calls for the development, of a precise and exact language which is used for the formulation of problems and arguments in mathematics and other objects. One of the important functions of the propositional calculus is to provide such a language. A familiarity with the laws of propositional calculus helps one to symbolise and formulate statements precisely and to derive their logical consequences, as is often necessary in mathematics.

By a proposition or statement, we mean an assertive (or declarative) sentence which can be classified to be as either true or false. We shall use small letters p, q etc. to denote propositions. Before proceeding with it, it will be interesting to look at some examples of propositions.

Examples: 1. Ram is a good student.

2. $2+3=5$
3 Today is very hot
4. Sugar causes tooth decay
5. If Shyam plays in our football team today, then our team will win.
6. Two angles of the plane triangle ABC are equal if and only if the sides opposite to the equal angles are equal.

Such examples can be readily multiplied.

As an example of an assertive sentence, which is not a proposition (i.e. which cannot be classified as true or false) we can cite the following: This sentence is false.

If it is the case that the above sentence is false, then it must be true and, on the other hand, if it is true then it is false by virtue of statement itself.

Truth values: If a proposition p be true, we give p the truth value T and if it be false, we give it the value F. From propositions p, q etc., we can form a new proposition by the use of connectives: 'not', 'and', 'or'; 'if' 'then' and 'if and only if'. Although it should be noted that 'not' is more of a modifier than a connective, we shall consider this to be one of the connectives.

Negation: To deny the truth of a statement, what do we do? We assert its negation. To contradict the statement 'Ram is a liar', we assert the following 'Ram is not a liar'. The negation of a statement p is denoted by $\sim$ p and $\sim$ p is false (true) when and only when p is true (false). The truth values of p and $\sim$ p are related as in the table below:

p	$\sim p$
T	F
F	T

Fig. 28

Let us now consider the other connectives. Consider the following statements:

(*i*) p = Ram goes to school
(*ii*) q = Shyam goes to school.

Then p and q (to be denoted by $p \wedge q$) and called the logical product or conjunction of p and q is the statement 'Ram and Shyam go to school'. p or q (to be denoted by $p \vee q$) and called the logical sum or disjunction of p & q is the statement 'either Ram or Shyam goes to school'.

If p then q (to be denoted by $p \Rightarrow q$ and called the implication of q by p) is the statement 'if Ram goes to school, then Shyam goes to school'. We also say that p is sufficient for q (or q is necessary for p). p is called the antecedent and q is called here the consequent. Let us take another illustration:

(*i*) p = the quadrilateral is a rhombus
(*ii*) q = this quadrilateral has perpendicular diagonals.

Then $p \Rightarrow q$ means that if this quadrilateral is a rhombus, then the diagonals of this quadrilateral intersect at right angles. This is evidently true. But can we say $q \Rightarrow p$? For we can have a quadrilateral which has perpendicular diagonals but which is not a rhombus. $q \Rightarrow p$ will be false if this quadrilateral be of the latter type This leads us to consider when $p \Rightarrow q$ and $q \Rightarrow p$ simultaneously hold. p if and only if q (to be denoted by $p \Longleftrightarrow q$) is the proposition $(p \Rightarrow q) \wedge (q \Rightarrow p)$. This is often expressed as the p is equivalent to q or p is a necessary and sufficient condition for q. As an example of this, we take the following example

p = this quadrilateral is a parallelogram.
q = this quadrilateral has pairs of equal opposite sides.

From elementary geometry it follows $p \Rightarrow q$ and $q \Rightarrow p$ are both true, i.e., $p \Longleftrightarrow q$ holds in this case. The sentence of the type $p \Rightarrow q$ is called conditional sentence while $p \Longleftrightarrow q$ is called a biconditional one.

How do the truth values of these sentences $p \wedge q$, $p \vee q$ etc. (which are called the compound sentences) depend on the truth values of component parts p & q?

It is clear $p \wedge q$ (i.e., p and q) is true when and only when p is true and q is true. p or q is true when one of the components (or both) is true. $p \Rightarrow q$ intutively means that q follows from p. Then $p \Rightarrow q$ can be false only when p is true and q is false. Then $p \Longleftrightarrow q$ which has been defined as $(p \Rightarrow q) \wedge (q \Rightarrow p)$ is thus true, if and only if p and q have the same truth values. Thus the truth table of these compound sentences can be given as:

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Longleftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	F	F	T	T	F
F	F	F	F	F	F

Fig. 29

A little thought will reveal that the values regarding the truth or falsity of the compound sentences follow the usual connectiveness of our everyday language.

Tautologies (*Logical laws*): In general, the truth value of a compound sentence depends on the truth values of component sentences. But there are certain cases where the truth value of a compound sentence is independent of the truth values of its components. Let us consider a few examples:

(*i*) $p \vee q \sim p$
(*ii*) $p \wedge q \Rightarrow p$
(*iii*) $p \Rightarrow p$
(*iv*) $p \Rightarrow q \Longleftrightarrow \sim p \vee q$
(*v*) $p \Rightarrow q \Longleftrightarrow \sim q \Rightarrow \sim p$
(*vi*) $p \Longleftrightarrow \sim \sim p$

Let us now consider the examples in details. Let p be the proposition "moon is a green cheese". Then $p \vee \sim p$ is the statement either moon is a green cheese or it is not This is obviously always true. It is clear instead of the above proposition that we could take any other proposition for p. Even then $p \vee \sim p$ will be always true.

Let us consider (*v*). Let p = Ram is an Indian, q = Ram can be trusted. $p \Rightarrow q$ will be "If Ram is an Indian, then he can be trusted". In this case $\sim q$ = Ram cannot be trusted, $\sim p$ = Ram is not an Indian, $\sim q \Rightarrow \sim p$ will be read as: If Ram cannot be trusted, then Ram is not an Indian. It can be seen easily that these assertions are essentially the same, i.e., one is true if the other is true.

Whatever be the truth values of p & q, the above propositions (compound sentences) are always true. Such propositions are called tautologies (or logical laws).

These tautologies are often used to prove mathematical theorems. Let

p = two angles of this triangle are equal
q = pair of two opposite sides are equal

An elementary proposition of school geometry is to prove: $p \Longleftrightarrow q$. But how do we usually go about proving it?

If possible, let the sides be not equal i.e., $\sim q$ holds and we arrive at the conclusion that $\sim p$ holds. Thus $\sim q \Rightarrow \sim p$ and hence we infer from (*v*) that p = q holds. Thus (*v*) embodies the principle of the proof by *reductio ab absurdum.*

If a biconditional sentence is a tautology then it is not necessary to distinguish the antecendent & consequent. Thus p & $\sim p$, $p \Rightarrow q$ and $\sim q \Rightarrow \sim p$ are essentially the same statements. Such propositions are said to be equivalent.

Variables and Open Sentences

We next introduce the concept of a 'variable'. Its very name suggests something that varies. More precisely, a variable is a symbol (usually we will denote it by x) that holds, a place for the name in a given set of elements. The given set of elements is called the domain or the replacement set.

Consider the following sentences:

(*i*) A flower x is red

(*ii*) $x+5=9$

(*iii*) $x+5>9$

The sentence (*i*) contains the variable. The replacement set can be taken to be the set of all flowers. We may note that the statement is true for some flower (which are red) while it is false for others. Similarly, in (*ii*) and (*iii*), if we take the replacement set, the set of positive integers then (*ii*) and (*iii*) are true for some positive integers while they are false for others. Thus, none of these sentences can be classified as either true or false. Such sentences are called open sentences (while the proposition can be called as closed sentences) The subset of the replacement set for which an open statement is true is called the truth set of the open sentence. For example, in (*i*) the truth set is the set of all red flowers. In (*ii*) the truth set is $\{4\}$ while in (*iii*) the truth set $\{5, 6, 7, \ldots\ldots\}$

It should be noted that by substitution of a particular value of the variable in an open sentence, we get a proposition. For example, in (ii) if we take $x=3$, $3+5=9$ is a proposition though it is false.

4.1. *Sentences*: Before beginning with this article, the teacher might ask the students that how many types of sentences they use in common parlance. The teacher will cite certain examples and ask his students to do the same. The teacher will then point out that these sentences are nothing but assertive (or declarative), interrogative, exclamatory etc. statements. He will then mention some statements that occur in mathematics, for example (i) sum of the three angles is equal to two right angles, (ii) $6>5$ etc. In mathematics, we call such statements to be mathematical sentences. The teacher may refer to the close analogy between a mathematical sentence and an English sentence. While in English sentences the subject is word or group of words that stand for a thing, a person, a place or even an idea and the predicate is a word or a group of words involving a verb that tells about the subject. In mathematics, the subject is usually a number, for example, in the sentence 6 is greater than 5, subject is the number and the predicate is 'is greater than'. The teacher will also mention that in a mathematical sentence, we use the symbol and this means that, in the above sentence, the predicate is "5". Further, the teacher will explain, through a number of examples, that a mathematical sentence can be of declarative type only. In otherwords, in a mathematical sentence, we can assert or declare categorically whether a sentence is true or false. For example, the sentence $6 \ngtr 5$ is a false one while $4+2=6$ is true. In analogy to the English sentences, we use in mathematical sentences the symbol '$=$', '$\neq$', '$<$', '$\leqq$', etc. as verbs which give a relation between two numbers. Incidentally, the teacher may remark that the group of words 'is greater than', 'is equal to', 'is not equal to' etc. may be called verb forms. If we use the verb 'is equal to' in a mathematical sentence, for example, $4+2=6$, we say that '6' and '$4+2$' are two of the many numerical (names) for the number 6.

Concepts: Mathematical sentences. subconcepts
Subconcepts : True and false sentences.
Vocabulary: Mathematical sentences, true and false sentences.

4.2. *Open Sentences*: Having explained the sentences, the teacher will proceed to take up examples with the objective of enlarging the variety of relations between numbers, which the mathematical sentences can express While considering some of the sentences, the teacher may take up a sentence of the Obviously, this sentence can be made declarative by varying the values of x and so the number x is called the variable. A sentence of this type is called an *open sentence* which may be either true or false but cannot be both. The variable cannot be chosen arbitrarily but it can be chosen from a given set of whole numbers which is sometimes called domain or range or replacement set of the variables.

Concepts: open sentences, variable, Domain, Replacement set.
Vocabulary: Open sentences, variable, Domain, Replacement set.

4.3. *Solution sets of open sentences*: The teacher will recall from the earlier article that an open sentence can be true only for a set of values of the variable from a prescriped range. By considering an open sentence with numbers, the teacher may draw up a table to show the true and false statements according to the nature of a variable. The values of the variable may now be taken as elements to form a new set which is called the truth set or the solution set. He may recall some notions of sets to facilitate his teaching.

The teacher will recall the number ray discussed earlier and ask his pupils to represent the members of the solution set and thus they will understand what is called the graph of the solution set of an open sentence.

Concept: Solution set.

Vocabulary: Solution set, Graph of solutions sets.

4.5. *Sentences with other connectives*: Having dealt with the sentences involving the verbs and $\neq$ the teacher would consider the examples with other symbols such as '$>$', '$<$' etc. which are also known as connectives. In the last article, the teacher will bring in the ideal of open sentences with these connectives toogether with the graphs.

Concepts: Connective

Vocabulary: Connective

Symbols: '$>$', '$<$', '$\ngtr$', '$\nless$' '$\leqq$', '$\geqq$'.

4.6. *Implicating:* Before introducing the 'implication', the teacher will cite a number of examples drawn from our daily experiences. The students will then be led to understand what that verb 'imply' means. Similar examples may be quoted to get accross the students what are meant by 'imply' and 'is implied by'. After these familiar examples, the teacher will take up mathematical sentences to explain the above connectives.

Concept: Implication

Vocabulary: Implication

Symbols: $=>$ (implies); $\Longleftrightarrow$ (Implies and is implied by)

4.7. *Equations and Solution sets*: An open sentence with '$=$' as the connective is called an equation. For example $x+4=9$ is an equation. The teacher should now tell his students to discover a number of examples of equations and how they possess the reflexive, symmetric and transitive properties. Moreover, that the additive and the multiplicative properties hold in respect of these equations should be explained to the student with copious examples. It is largely in connection with equations, that we use the term 'solution-set' which is the truth set of the open sentence. Therefore, any value of the variable i.e. any number of the solution set of the equation (which is an open sentence) is a solution or a root of the equation. The teacher will now illustrate the above properties of equalities in finding out the solutions of equations.

Concepts: Equations and solutions set of Equations.

Vocabulary: Equations, Roots of Equations, Solution set of Equations.

III. Understanding and Skills to be Achieved

As the topics here are meant for use of a correct and a precise mathematical language, the teacher should see that his students should pick up the appropriate ones. The students should understand properly what are meant by (i) an open sentence and (ii) a truth set.

A multitude of exercises will enable the students to acquire the techniques of finding the truth set, the graph of a truth set and the solution of some simple questions.

VI. Class-room and Student Activities

As this chapter is essentially a continuation of the study of numbers, the teacher should provide his students with questions and examples so that they can discover the properties by themselves.

VII. List of Concepts, Vocabulary, Symbols

Concepts: Mathematical sentences, open sentences, variable,, domain, replacement sets, solution set, connective, implication, equation, solution set of equation.

Sub Concepts: True and false sentences.

Vocabulary: Mathematical sentences, true and false sentences, open sentence variable, domain, replacement set, solution set, graph of solution sets, connective, implication, equations, roots of equations, solution sets of equations.

Symbols: '$=$' '$\neq$' '$>$', '$<$' '$\ngtr$' '$\nless$', '$\geqq$', '$\leqq$' '$\Rightarrow$', '$\Longleftrightarrow$' '$\wedge$' '$\vee$', '$\sim$'.

CHAPTER V

APPLICATIONS—SIMPLE CASES OF DIRECT VARIATION.

1. Aims and Objectives of the Chapter

The chief purpose of this chapter is to provide the students with applications of the number systems that they have learnt in the earlier chapters. These applications have been so designed that they are highly relevant to the real experiences of the students in daily life. In otherwords, an acquaintance with what has been dealt here will convince the students of the use of algebra or, strictly speaking, algebraic formulae in the solution of some of the difficult problems of arithmetic such as variation, ratio and proportion, simple interest, work and time and so on and the only advantage in the use of such formulae is that they help in obtaining an unified treatment of the apparently disparate classes of examples considered hitherto in isolation from each other.

2 Methods of introducing Concepts

General notes: The topics—ratio and proportion—are so related to the topics that have been taken up in the book that it would be worthwile to begin with what we mean by 'ratio' and 'proportion'. Undoubtedly, these form a familiar part of everyday experiences of the students

We use the word 'ratio', when we have to compare two (or more) numbers or quantities and a ratio of two numbers is expressed as the quotient obtained by dividing one of them by the other. On the other hand, a proportion denotes an equality of ratios.

Specifically, to compare ratios, we associate with each ratio a fraction, for example, in water of 1500 gms. there are .30 gms. of lime to the ratio .30 to 1500, we associate

$$\frac{.30}{1500} = \frac{1}{5000}$$

If we find that the equivalent fractions of two ratios are equal, we say that the ratios are equal. It may be observed that the numbers in a ratio are *ordered*, for example, when we say that the rod A is twice as long as the rod B, we express it by the ratio as 2:1 and certainly, the meaning would be different if the ratio of the length of B to that of A is considered, in which case it is 1:2. As regards an example of proportion, we might compare the scale with the slide rule and may find that the set 1, 2, 3, 4, 5....and the set 2.2, 4.4,....are certainly connected. We then say that the elements of the two sets are *proportional*. We can also say that the ratios of the corresponding elements of these two *ordered* sets is constant. It can be proved that the product of the two extreme terms of any proportion is equal to the mean terms of the same proportion. This means if, a, b, c, d are in proportion i.e. $a : b :: c : d$ then $ad = bc$

It can be seen easily that

$$a = \frac{bc}{d}, \quad d = \frac{bc}{a}$$

$$b = \frac{ad}{c}, \quad c = \frac{ad}{b}$$

It may be mentioned that the concept of proportion is of immense use in geometry also. Let us now take a deeper look into it. When we want to write a ratio as a means to compare objects in two sets, we really express a relation between number of objects in the two sets and, in essence, it is a relation, which is, nevertheless, an important concept in abstract algebra and we may now refer to the relevant supplementary material for this. An important relation which we often use in our daily life largely pertains to the idea of variation. This can be thought of in the following ways:

(1) When one object increases, another object may increase.
(2) When one object decreases, another object may decrease.
(3) When one object increases, another object may decrease and vice-versa.
(4) When one object increases, or decreases, a number of other objects may increase or decrease.

The first two of the above relations may be put in other ways. One quantity A is said to *vary directly* as another B, when the two quantities depend upon each other in such a manner that if B is changed, A is changed in the same ratio. The third relation which we call an inverse relation signifies *inverse variation* or 'varies *directly* as the reciprocal of. The fourth relation which we call '*joint variation*' may be stated as 'one quantity is said to vary jointly as a number of others when it varies directly as their product.'

Thus, A varies jointly as B and C when $A = mBC$ where m is a constant. For instance, the interest on a sum of money varies jointly as the principal, the time and the rate per cent. A varies as B/C means A varies directly as B and inversely as C.

It may so happen that the variation of A depends partly on that of B and on that of C. Let us suppose these variations take place separately, each in its turn producing its own effect on A; also let a, b, c be a set of simultaneous values of A, B, C.

1. Let C be constant while B changes to b; then A must undergo a partial change and will assume some intermediate value a′, such that

$$\frac{A}{a'} = \frac{B}{b} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

2. If C changes to c when B is constant, that is, B retains its value b; then A must complete its change and pass from its intermediate value a′ to its final value a, where

$$\frac{a'}{a} = \frac{C}{c} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From (1) and (2), we find

$$\frac{A}{a'} \times \frac{a'}{a} = \frac{B}{b} \times \frac{C}{C}$$

i.e.

$$A = \frac{a}{bc} \cdot BC$$

or A varies as BC.

'Ratio', as stated above, is a comparison of two quantities and these quantities must be expressed in terms of the same unit. Thus, the ratio of Rs. 2 to .75 P. is measured by the fraction $\frac{2 \times 100}{75} = \frac{8}{3}$

This is read as 8 is to 3 and written in the form 8:3. The ratio of A to B is usually written as A:B. The quantities A and B are called the *terms* of the ratio. The first term is called the *antecedent*, and the second term, the *consequent*.

Now, $\frac{p}{q} = \frac{mp}{mq}$ and thus the ratio p:q is equal to the ratio mp: mq; so that the value of the ratio remains unaltered when the antecedent and the consequent are multiplied or divided by the same quantity. It can also be shown that two or more ratios may be compared by changing their equivalent fractions to a common denominator. Let us take two ratios p: q and x:y.

Now, $\frac{p}{q} = \frac{py}{qy}$ and $\frac{x}{y} = \frac{qx}{qy}$, hence the ratio p:q, is greater than, equal to, or less than the ratio x:y according as py is greater-than, equal to or less than qx.

Let the ratio $\frac{p}{q} : \frac{r}{s}$ be measured by the fraction $\frac{p/q}{r/s}$ or $\frac{ps}{qr}$; this is equivalent to the ratio ps:qr Thus, the ratio of two fractions can be expressed as a ratio of two integers. We may remark in passing that if the ratio of any two quantities can be expressed exactly by the ratio of two integers, the quantities are called commensurable; otherwise, they are incommensurable. When the ratio x:y is compounded with itself, the resulting ratio $x^2 . y^2$ is called the duplicate ratio. Similarly, $x^3 \cdot y^3$ is the triplicate ratio, and $x^{\frac{1}{2}} : y^{\frac{1}{2}}$ is called the subduplicate ratio of x:y.

Let us take a ratio r-s or $\frac{r}{s}$ and let $\frac{r+z}{s+z}$ be the new ratio obtained by adding z to both terms.

$$\text{Now, } \frac{r}{s} - \frac{r+z}{s+z} = \frac{rz-sz}{s(s+z)}$$
$$= \frac{z(r-s)}{s(s+z)}$$

Here, r − s is positive or negative according as r is greater or less than s.

Thus, if $r > s, \frac{r}{s} > \frac{r+z}{s+z}$

If $r < s ; \frac{r}{s} < \frac{r+z}{s+z}$

So, we may say, that a ratio of greater inequality is diminished and a ratio of lesser in equality is increased by adding the same quantity to both the terms. Similarly, it can be shown that a ratio of greater in equality is increased, and a ratio of lesser in equality is diminished, by subtracting the same quantity from both its terms.

Let us take a number of equal ratios such as

$$\frac{\alpha}{\beta} = \frac{\gamma}{\delta} = \frac{\xi}{\eta} = \dots\dots\dots\dots$$

It can be proved that each of these ratios is equal to

$$\frac{p\alpha^n + q\gamma^n + r\xi^n + \dots\dots\dots}{p\beta^n + q\delta^n + r\eta^n + \dots\dots\dots}$$

where p, q, r, n are any quantities whatsover. Many important results can be obtained from this general proposition, by giving different values to p, q, r, n. For example, if $\frac{\alpha}{\beta} = \frac{\gamma}{\delta} = \frac{\xi}{\eta} = \dots$

then each of these ratios $= \frac{\alpha + \gamma + \xi + \dots}{\beta + \delta + \eta + \dots}$

5.1: *Introduction:* To drive home the ideas envisaged in this article, the teacher may begin with numerous examples drawn from the social life of a city, of a town or of a village. The examples in the book are essentially those which everyday life offers in respect of proportional magnitudes. The students should be asked to build more tables of proportional magnitudes. It is through these examples, the teachers will encourage the students to write the relationships in the form $y/x=1$, $y/x=2$, $y/x=3$ *which are called the constant ratio* (or ready-reckoner relationship). Similarly, the teacher will introduce the constant product relationship such as $xy=k$ (constant).

Concepts: Directs variation; inverse variation.

Vocabulary: Proportionality.

5.2; 5.3, 5.4 *Interest and principal etc.* The concepts or 'interest and principal' 'interest and time', 'interest and rate' or simply 'simple interest' can be introduced in interesting ways. The foregoing relationship may be used to introduce the concept of simple interest. We may begin with an example. A person lends Rs. 200 to a needy man under the condition that he has to pay back at the end of each year a sum of Rs. 5% in excess. Mathematically speaking, Rs. 200 is called the '*principal*', Rs. 5%, '*the rate of interest*', and the *time* in the number of years, and the total money paid back by the borrower is the *amount*.

Thus, Amount = Principal + Interest or in short, $A = P + I$
Two more formulas can be obtained from this.

(1) $A - P = I$
(2) $A - I = P$

The interest is calculated in two ways:

(i) at the expiry of the total time.
(ii) at uniform periodic intervals (say, half-yearly or yearly) and it is added to the principal after each interval of time.

The interest calculated by the first method is called the 'simple interest' (S.I.) and the second method is used for calculating 'compound interest'. In this class, we are concerned with S.I. only. This can be calculated easily by the formula

$$\text{S. I.} = \frac{P \times r \times t}{100}$$

where, P denotes principal, r stands for rate percent per year, t for the time of years.

The remaining topics may be introduced as shown in the book.

Concepts: Simple interest, rate percent, principal, amount.

Vocabulary: Simple interest, principal, borrower, lender, interest, rate of interest, rate percent, time and work, quantity, cost.

3. Understanding and skills to be achieved

The student should understand the varous relationships between the different variables through a number of experiments conducted in the class room and in particular, the idea of proportionality should be understood carefully The teacher should be very careful to see the fact that examples given to the students for calculating simple interest may easily be performed with the help of formulas. Sufficient opportunity should be given to the students so that they may use two main formulas

(i) $A = P + I$
(ii) $\text{S.I.} = \frac{P \times r \times t}{100}$

4. Class room and student activities

The slant on the teaching of these topics should be based on appeal to examples drawn from the everyday life. Some models may be given to the students, for example, square cardboards to get across them how the words 'ratio' and 'proportion' occur very naturally. It may be helpful to explain the proportionality by means of pouring a pint of water, or a gallon of water etc. in a tumbler. It may be highly profitable if the students are assigned to work in different groups. The idea of variation which is nothing but relationship may be explained to the students by graphical representations that appeal to children of all abilities. A detailed study of graphs might enable the students to discover in the classroom, various relationships and will eventually lead them to discover the what of proportionality.

5. List of concepts, vocabulary

Concepts: Proportionality
Vocabulary: Proportionality, rate percent, amount, interest, simple interest, principal.

KELLEY may be cited and the central feature of each of these books is that they would lead the readers to abstract ideas through the discussion of concrete situations, often pointing to the errors that we often gloss over in respect of some topics taught at the school level such as factorization of polynomials, solution of polynomial equations etc.. For the theory of sets, one may start off with VILENKIN. The next two books to be read may be KAMKE, HALMOS and finally, a teacher may delve into deeper axiomatic treatment of set theory by SUPPES and by BOURBA KI. Even though the reading of the books by HERSTEIN and LANG, which bristle with interesting and copious examples, would enable the readers to progress at a very fast rate, they can have an opportunity to have the beautiful glimpse of the subject only through the masterly treatises on algebra such as those by HASSE who is considered to be 'one of the founding fathers in abstract algebra', BOURBAKI, VAN DER WAERDEN, ALBERT, BIRKHOFF & MACLANE, WARNER, JACOBSON, GODEMENT. Although the books by VAN DER WAERDEN and by BIRKHOFF & MACLANE are held to be classic ones, they are still very modern in outlook on many points, not to speak of the exemplary style which is so characteristic of master minds.

As the textbooks will show, the idea of sets has been used to develop the algebraic properties of the system of whole numbers integers, rational numbers and real numbers. For a thorough treatment of this development, one has to consult the book by PATERSON & HASHISAKI. But, if one is to mention the appropriate books in which the teachers may get treatments at a deeder level, one must quote first the scholar treatise by LANDAU while the books by NIVEN, THURSTON, BELL may be taken up next. Again, as the theory of numbers constitutes an important section in the textbooks, we have to cite the books by HUNTER, NIVEN & ZUCKERMAN, and certainly, the seminal books by DICKSON and HARDY & WRIGHT. Some books on group theory have been enlisted, keeping in view the fact that they would not merely enrich the equipment of teachers' but would supplement their teaching in geometry. These books are those by LEDERMAN, PAPY, BELL & FLETCHER, MANSFIELD & BRUCKEN-

HEINER, GROSSMAN & MAGNUS. Besides these, it is essential that the teachers should keep themselves aware of the curricular materials (including students' texts and teachers' guides) produced by a number of projects in other parts of the world Of these, those by SCHOOL MATHEMATICS PROJECTS (S.M.P.), MIDLANDS MATHEMATICAL EXPERIMENT (M M.E.), SCOTTISH MATHEMATICS GROUP, SHROPSHIRE MATHEMATICS EXPERIMENT,—all of U.K., SCHOOL MATHEMATICS STUDY GROUP (S M.S.G.), UNIVERSITY OF ILLINOIS COMMITTEE ON SCHOOL MATHEMATICS (UICSM), SECONDARY SCHOOL MATHEMATICS CURRICULUM IMPROVEMENT STUDY (SSMCIS),—all of U.S.A., NATIONAL COUNCIL OF EDUCATIONAL RESEARCH & TRAINING (NCERT) of our country deserve mention. A glance through these materials as well as some books on new mathematics at the secondary level such as those by MARJORAM, MATHEWS, GRAY, and others would be very helpful in teaching these topics at the school level.

The approach to the teaching of new mathematics in the school has, doubtless, to be a novel one, at least different from what have been followed so long. In otherwords, the padagogical and psychological principles have necessarily to be reckoned with, if the teaching is to be meaningful.

In this connexion, one can hardly ill afford to mention the relevant books, one by the great psychologist PIAGET and the other, by DIENES and the teachers may also read SKEMP'S textbooks based on these principles.

It is often found that the teaching can be made highly interesting if it is accompanied by references to a few historical anecdotes and it is with this aim that the books on history of Mathematics by STRUIK, BELL, CAJORI, DATTA & SINGH, KLINE and on men of mathematics by BELL, AABOE. In this context, the book by BOURBAKI deserves a special mention, for, it seems, no book can surpass this book so far as the finesse in presentation of historical facts is concerned. The interest in the students can be further stimulated, if the teachers can provide the students with a host of mathematical recreations and the books by ROUSE BALL, RADEMACHER & TOEPLITZ, SINGH, STEINHAUS, STEIN, CUNDY & ROLETT, BERGAMINI seem to be pertinent to the purpose. As for problems, the books by STEINHAUS, SHKLANSKY *et al*, YAGLOM, SIERPINSKI have been suggested.

It is felt that the teachers should be well informed of the educational researches not only in their fields of interest but in all the topics that have relevance to what they teach in schools. Accordingly, some literature brought out by O.E.C.C., UNESCO, COMMONWEALTH SECRETARIAT, to mention a few have been added to the list. It remains yet to refer to journals which contain articles on mathematics and mathematics education, and which, therefore, mirror recent developments in the subject; for, a successful teacher has to be regularly in touch with them, otherwise, the teacher may run the risk of falling out of step with the tremendous pace with which the innovations in school mathematics are proceeding.

APPENDIX II

SOME SUGGESTED BOOKS AND JOURNALS FOR STUDY AND REFERENCE PURPOSES

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ERRATA

Page	*Line from the*		*For*	*Read*
	top	*bottom*		
V		2	NCRET	NCERT
11		2	For an empty set	For the empty set
16		2	will 'no'	will be 'no'
27		15	numbers where	numbers were
29	18		y, x, y ε S	y; x, y ε S
32		12	G Peana (1889—)	G. Peano (1889—1932)
35	4		Italian Bombelli	Italian Mathematician Bombelli
35	7		exponents is	exponents were
35	18		{1, 2, 3, 4, 6, 8, 12, 24	{1, 2, 3, 4, 6, 8, 12, 24}
37	14		$2^{2^n} + 1$	$2^{2^n} + 1$
38	2		Frienderich	Friedrich
38		4	E C F	H. C F
28		11	{1, 2, 3, 4, 6, 12}	i e {1, 2, 3, 4, 6, 12}
39	2		D (60) {1, 2, 3, 6}	D (60) = {1, 2, 3, 6}
39	14		esaily	easily
39	18		x EN	x ε N
40		5	there are	are there
40		8	statement	statement,
41	17		articles	article
42	5		For the absence	In the absence
43		15	dealt	deals
45	13		ask more	ask for more
45	17		earlier section	earlier sections
45		4	meaning	meanings
45		1	We get B	We get B,
46	1		complement	complements
48	12		when is it possible?	when it is possible
48	18		by 'a' and itself	by 'a' and 'a' itself
48	18		a 0 = 0 Therefore, no set	a . 0 = 0, no set
50		14	Vocabulary.	Vocabulary:
52		4	to develop in	to develop
55	14		The teacher nothing may	The teacher may
61	11		Egyption	Egyptians
63		12	(say)	(say),
63		2	exponent	exponents

Page	*Line from the*		*For*	*Read*
	top	*bottom*		
94		4	10 10 1	100 10 1
76	14		the truth set	the truth set is
77		18	sentence of the	sentence of the type $x+3=12$
78	17		introd ucing	introducing
78		5	student	students
78	4		solutions set	solution set
80	10		wofk	work
83		12	$x\frac{1}{2} . y\frac{1}{2}$	$x^{\frac{1}{2}} . y^{\frac{1}{2}}$
84		12	Directs	Direct
84		10	concept or	concept of
86		3	concepts	concept
88	16		algebra',	algebra, vide,
88		10	deeder	deeper
88		10	scholar	scholarly
88		3	teachers'	teachers
89	1		HEINER	HEIMER
89	5		PROJECTS	PROJECT
89	12		(all of U.S.A)	all of U.S.A
89	19		the school	schools
92		21	Introduction t	Introduction to
92		5	'Teachers' Guide	Teachers' Guide
93		12	Fransisco	Francisco
93		5	Snspshots	Snapshots
94	1		Mc Graw	McGraw
94	6		Yofk	York
94	11		Their	their
94		11	Mathematics	Mathematics,
94		4	Shihshak	Shikshak
94		1	American Mathematical	American Mathematical monthly